Methods of representation theory with applications to finite groups and orders,
   vol. 1, by Charles W. Curtis and Irving Reiner, Wiley, New York; Chiches-
   ter, Brisbane and Toronto, 1981, xxi + 819 pp., $55.00.

At last! After 20 years, C & RII, or at least its first volume, has appeared. Its
predecessor, [C & RI] which was otherwise known as *Representation theory of
finite groups and associative algebras*, had become something of a classic since
its publication in 1962. It has been widely used as a text and general reference
on representation theory. *The Scientific Citations Index* lists over 600 journal
articles that cite it as a reference. It differs from other books in the field in that
it attempts to cover all aspects of representation theory. Few volumes on group
representations contain more than a minimal amount of the integral theory,
while most books on integral representations use little material from the
ordinary or modular theory. One of the appeals of [C & RI] is that it considers
both. But because of the extensive progress in these areas over the last two
decades, a revision of [C & RI] was appropriate.

However, let us make no mistake. Any suggestion that the new volume is
merely a revision or up-dating of the old would do the book a grave injustice.
In many ways the two are similar. The styles of writing and organization are
essentially the same. But Curtis and Reiner have reorganized the subject matter
considerably. They take pains to use different proofs and more modern
approaches to theorems wherever appropriate. They make greater (though not
extensive) use of homological techniques and include many results that have
been proved since the publication of [C & RI]. The major emphasis of C & RII
is representations of finite groups, although much material on the structure
and representations of algebras and orders is also included. The several
group-theoretical results in the book are proved so as to illustrate the power of
the representational methods. While a few topics from [C & RI] have been
omitted, the treatment of many others has been expanded. It is, of course,
impossible to cover all of representation theory in one or two books. However
given the length of C & RII and the fact that a second volume is to appear
soon, it is difficult to take very seriously the authors' contention that their
approach is "not intended to be encyclopedic".

This book is not a research volume in the sense that it contains few if any
new results. Rather it is intended to provide a basic background in representa-
tion theory. Many areas of greatest research interest will be treated only in
volume 2, and several sections in this first volume appear to be aimed at
preparation for the next. The book begins with an introduction consisting of
nearly 200 pages of preliminary material. Following the introduction, the first
two chapters discuss primarily the theory of ordinary and modular group
representations and their characters. The main thrust of the book is the

discussion in Chapters 3 and 4 of integral representations and the more general theory of lattices over orders. This is the area to which Irving Reiner has contributed so much over the years and several of his results are included. In total the integral representation theory accounts for almost half of the book, excluding the introduction.

Integral representation theory is the bridge between the modular theory and the theory of ordinary (or complex) characters and representations. We begin the usual approach to modular representations by taking a p-modular system (K, R, k) consisting of an algebraic number field K, an integrally closed discrete valuation ring $R \subseteq K$ with maximal ideal P, and the finite field $k = R/P$ of prime characteristic p where $p \in P$. We assume that p divides the order of the finite group G and that K and k are large enough to be splitting fields for G and all of its subgroups. Suppose that V is a finite-dimensional irreducible KG-module. In V we choose an RG-lattice M. That is, M is a finitely-generated free R-submodule of V which is closed under the action of G, and M contains a K-basis for V. Such a lattice can be found because of the restrictions that have been put on R (see §§16 and 23 of C & RII). Now by reducing modulo the prime ideal we obtain a kG-module $\overline{M} = M/PM$. In general $\overline{M}$ is not irreducible and is not uniquely determined. If N is another such RG-lattice in V then $\overline{N}$ need not be isomorphic to $\overline{M}$. However $\overline{M}$ and $\overline{N}$ do have the same composition factors and hence also the same characters. Therefore each irreducible K-character of G corresponds to a sum of irreducible k-characters. This correspondence defines a decomposition matrix which can be used to determine the blocks of characters of G.

There are several reasons for doing this. Many problems in modular representation theory are susceptible to various types of local analysis, and often information obtained by these methods can be translated back to give results about K-representations of a group. The local analysis consists usually of relating the representations of kG to those of kH where H is the normalizer of some p-subgroups of G. Examples of such are the Green correspondence (see §20) and the Brauer correspondence of block (which will be treated in volume 2).

Why does this happen? Much of the grandure of the Brauer theory of modular representations and blocks and much of the difficulty of integral representation theory can be viewed as consequences of a single fact: the group ring RG is not a maximal order in KG. It is an R-order in the sense that it is an R-algebra, it is finitely generated as an R-module, and $K(RG) = KG$. However, it is properly contained in other R-orders of KG. This can be seen by letting $\gamma = \Sigma_{g \in G} g \in RG$. The element $\beta = (1/|G|)\gamma$ is an idempotent in KG and is in a maximal order that also contains RG. But $\beta \notin RG$. The standard proof that kG is not semisimple is obtained by observing that the image of γ is in the radical of kG.

There is an extensive literature on maximal orders, and in many cases the structure of a maximal order can be determined and is quite nice. For example, suppose that $KG = \oplus_{i=1}^{n} A_i$ where each A_i is an indecomposable two-sided ideal. Here $A_i = KGe_i$, where e_i is a primitive idempotent in the center of KG

and $1 = e_1 + \cdots + e_n$. Then A_i is isomorphic to the ring $M_{t_i}(K)$ of all $t_i \times t_i$-matrices over K. A maximal R-order in KG is a direct sum $\Lambda = \bigoplus_{i=1}^n \Lambda_i$ where Λ_i is a maximal order in A_i (Theorem 26.33). But $\Lambda_i \cong M_{t_i}(R)$ and $\Lambda_i/P\Lambda_i = \Lambda_i/\text{Rad } \Lambda_i \cong M_{t_i}(k)$. Consequently, if RG were a maximal order, then $kG = RG/(P \cdot RG)$ would be semisimple. For an irreducible KG-module V, the correspondence $V \to \overline{M}$, described earlier, would always yield an irreducible kG-module. Block theory would reduce to a triviality.

For the person studying integral representations, the fact that RG is not a maximal order causes numerous problems. One of the most striking is that of representation type. If Λ is a maximal R-order then we say that Λ has finite representation type because the number of isomorphism classes of indecomposable Λ-lattices is finite (see §33). But the group ring RG is always of infinite type whenever a Sylow p-subgroup of G is either noncyclic or has order greater than p^2. In 1962 Alex Heller and Irving Reiner gave one of the original proofs that the group ring ZG has an infinite number of nonisomorphic indecomposable modules when G is cyclic of order p^3 [3]. They showed first that it is only necessary to consider Z^*G where Z^* is the p-adic integers. Then they proved that there are two Z^*G-lattices M and N such that for every positive integer t, there exists an indecomposable lattice Y_t which is an extension of $N^{(t)}$ by $M^{(t)}$. Here $M^{(t)}$ is the direct sum of t copies of M. Hence there are indecomposable ZG-lattices of arbitrarily large Z-rank. The proof given in C & RII relies on a theorem of Dade (Theorem 33.8) that concerns more general nonmaximal orders.

There are many other problems that are encountered in integral representation theory, but that do not arise in the ordinary or modular theory. For example, in only a few cases does there exist an analog to the Krull-Schmidt Theorem for lattices over orders. One consequence of a theorem of Roggenkamp (Theorem 36.6) is that if G is commutative then there is a Krull-Schmidt type theorem for projective RG-lattices. But even this could fail if the ring R were only semilocal. In some cases cancellation of lattices in a direct sum is not a valid operation. Swan has produced the following example [5]. Let G be the quaternion group of order 8 and let I be the ideal in ZG generated by 3 and $\gamma = \Sigma_{g \in G} \, g$. Then $Z \oplus I \cong Z \oplus ZG$ but $I \not\cong ZG$.

In spite of this "bad news", integral representation theory has been and continues to be a very active area of research. The work has been spurred on by the discovery of many deep and powerful results. A sample list of such might begin with the classical Jordan-Zassenhaus Theorem (Theorem 24.1) which asserts that for any positive integer n, and for R-order satisfying certain nice conditions the number of isomorphism classes of lattices of R-rank less than n is finite. There is Maranda's Theorem (Theorem 30.14) which states that if R is a discrete valuation ring with prime element π, then the isomorphism class of a Λ-lattice M is determined by the class of $M/\pi^t M$ as a module over $\Lambda/\pi^t\Lambda$ for sufficiently large t. A basic result of Swan (Theorem 32.11) says that if R is a Dedekind domain whose field of quotients has characteristic zero, and if no prime dividing the order of G is a unit in R, then every finitely-generated projective RG-module is locally free.

I have mentioned only a few of the many impressive results on integral representations that are contained in C & RII. Other sections in Chapter 3 treat such topics as extensions of lattices, annihilators of Ext groups, maximal orders in group algebras and twisted group algebras and crossed-product orders. The section on maximal orders is primarily a list of results on their existence and structure. The reader is referred to Reiner's book [4] for most of the proofs. The authors do provide a complete proof that maximal orders are hereditary. The concentration in Chapter 4 is more on the structure of lattices over orders in algebras defined over local and global fields. The sections cover such topics as genera of lattices, projective lattices and their characters, Bass and Gorenstein orders, and the existence of Krull-Schmidt type theorems. §34 has many examples of integral representation some of which serve to illuminate the discussion of representation type in the previous section. The treatment of invertible ideals will be followed by a complete chapter on class groups in the second volume.

Chapter 1, which follows the introduction, covers the basic theory of representation and characters of finite groups. The main emphasis is on representations over fields of characteristic zero. It includes much of the standard materials on subjects such as the orthogonality relations of characters, induced modules, Clifford theory, and the Artin and Brauer induction theorems. In many cases the treatment has been changed or expanded from that in [C & RI]. Also several new wrinkles have been added. The section on Clifford theory has a discussion of Hecke algebras. Adams operators on the ring of virtual characters are included in the section on tensor algebras. There is a detailed discussion of tensor induction, the transfer and some of its consequences. The section on exceptional characters and special classes has, among other applications, a proof of the Brauer-Suzuki theorem on groups which have generalized quaternion Sylow 2-subgroups.

Modular representation theory has been elevated to a more prominent position in C & RII. In [C & RI] it had been relegated to a comparatively short chapter at the end of the book, and that chapter included a treatment of block theory. In the new edition block theory will be seen only in the second volume. Instead the concentration in Chapter 2 of C & RII is on modules and characters over modular group algebras. The usual relationship between the ordinary characters and Brauer characters is developed using the "Cartan-Brauer Triangle", a commutative diagram of Grothendieck groups two sides of which are the decomposition and Cartan maps. The chapter contains an account of vertices, sources and the Green correspondence, together with an application to the theory of exceptional characters. The authors give a proof of Green's indecomposability theorem that uses the techniques of graded Clifford systems. Other sections include induction theorems over arbitrary fields and a proof of the Fong-Swan-Rukolaine theorem concerning the liftability of simple modules over p-solvable groups.

I strongly recommend C & RII to anyone studying finite group representations. One of its strongest points is its wealth of examples and problems. The book should be particularly valuable to those interested in understanding the

ins and outs of the integral theory. I believe that this material is better presented than it was in [C & RI]. The topics are more clearly delineated and it is easier to understand the problems and motivation.

My one complaint with the book is that it does not really replace [C & RI], or more generally, that it is not entirely self contained. It was mentioned earlier that the fundamental material on maximal orders is largely quoted without proof from [4]. There are many other places, particularly in the introduction, where the reader is referred to [C & RI] or to [4] for the proof of a theorem. Admittedly most of these results are standard and can be found elsewhere, but in many cases no other references are given. The authors state in the introduction that whenever there was an "overlapping between this book and [C & RI] we have tried to always give new proofs or a revised presentation". This is certainly an admirable endeavor and they have to a great extent succeeded. Inclusion of the omitted would have significantly lengthened a volume which runs some 800 pages as is. I do not wish to overemphasize this point. For most readers it will have very little effect on the book's utility. Yet personally, I had hoped to remove my copy of [C & RI] to the math-history shelf.

It is anyone's guess as to whether C & RII will be as widely used as reference and text as its predecessor [C & RI]. In the areas of ordinary and modular representations it faces stiff competition especially with the recent appearance of the volumes by Blackburn and Huppert [1] and Feit [2]. For integral representations it gives the most extensive and general treatment available for the subjects that it covers. Of course the usefulness of the book will depend greatly on the contents of the second volume. I have not seen the second volume, but it promises to cover several topics in currently very active research areas. According to the introduction of C & RII the chapter titles will be

Burnside rings and representation rings,
Rationality properties of group representations,
Representations of finite groups of Lie type,
Indecomposable modules,
Blocks,
Algebraic K-theory,
Class groups.

Having read the first volume, I am eagerly awaiting publication of the second.

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