review, is supplemented by 236 pages of appendices that bring the book up-to-date, and provide a more systematic treatment of some topics from the French version. In summary, the authors have prepared a valuable reference for mathematicians and engineers.

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Bounded analytic functions, by John B. Garnett, Academic Press, New York, 1981, xvi + 468 pp., $59.00.

The book under review belongs to an area which, for want of a better term, I shall call one-dimensional function theory. "Function theory" should be interpreted here, not in the old sense of the theory of functions of a complex variable, but rather in a broader sense encompassing both the analysis of functions, holomorphic or not, and the analysis of spaces of functions. The settings for one-dimensional function theory are primarily the unit disk and the upper half of the complex plane together with their boundaries, the unit circle and the real line, respectively.

One-dimensional function theory is not a branch of mathematics in the way that, say, operator theory and low-dimensional topology are. Perhaps it does not even deserve a name of its own. The operator theorist seeks to understand the structure of operators, the low-dimensional topologist to understand the structure of three-dimensional and four-dimensional manifolds. The practitioner of one-dimensional function theory is aware of no comparable ultimate goal. This in part reflects the status of one-dimensional function theory as a handmaiden of several other, more coherent, disciplines—operator theory, theory of Banach spaces and topological vector spaces, prediction theory, systems theory, theory of commutative Banach algebras—which it provides with basic tools and examples. Much of the story of one-dimensional function theory is in fact the story of the interaction of complex analysis and harmonic analysis with these other disciplines. Nevertheless, whether or not the area deserves a name of its own, it does possess a soul of its own and a beauty all its own.

The theory began with the thesis of P. Fatou, published in 1906 [**1**]. Fatou established the almost everywhere existence of nontangential boundary values for certain classes of harmonic and holomorphic functions in the unit disk, setting the stage for the interplay between real and complex analysis. Ten years later, F. Riesz and M. Riesz, building on and extending Fatou's results and methods, presented their famous theorem on analytic measures: A nonzero complex Borel measure on the unit circle whose Fourier coefficients with negative indices vanish is mutually absolutely continuous with respect to Lebesgue measure [**2**]. The theorem implies, for example, that the boundary values of a nonzero bounded holomorphic function in the unit disk cannot vanish on a set of positive measure.

The spaces H^p were named by F. Riesz in a paper published in 1923 [**3**]. For $p > 0$, H^p is the class of holomorphic functions f in the unit disk that satisfy

$$\sup_{r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it})|^p\,dt\right)^{1/p} < \infty .$$

The "H" was in honor of G. H. Hardy, who, in 1915, had shown that the p-norms appearing in the above supremum increase with r [**4**]. Riesz established a fundamental factorization theorem, showing that an H^p function f can be written as bg, where b is a so-called Blaschke product and g is free of zeros. The function b is bounded with boundary values of unit modulus (a.e.) while g belongs with f to H^p. Riesz used his factorization theorem to obtain simple proofs of known and new results. This general line of investigation, which belongs to the theory of the boundary behavior of holomorphic functions, developed steadily in the 1920s and 1930s. At the same time another aspect of the subject, boundedness theorems, emerged and grew. The theorems of M. Riesz [**5**] and A. N. Kolmogorov [**6**] on the L^p and weak L^1 boundedness of the conjugation operator appeared in 1924 and 1925, respectively. Hardy and J. E. Littlewood introduced their maximal function and applied it to H^p functions in 1930 [**7**].

Since about 1950, one-dimensional function theory has been pervaded by the viewpoint of abstract analysis. This trend was signaled in 1949 by A. Beurling [**8**] who, using a refinement of F. Riesz's factorization (the inner-outer factorization), classified the closed invariant subspaces of the unilateral shift operator on the Hilbert space H^2. It became clear to mathematicians who had been reared on a diet of functional analysis that the spaces H^p and their near relatives offer substantial examples and problems to illuminate the general theory.

Here are three samples. 1. One can see from the F. and M. Riesz theorem that the Banach space H^1, unlike its father, L^1, is a dual space. Therefore, by the Krein-Milman theorem, the unit ball of H^1 has extreme points. What are they? 2. The functions that are holomorphic and uniformly continuous in the unit disk form a Banach algebra, called the disk algebra, under the supremum norm. Can one describe the closed ideals of that algebra? 3. The space H^∞ of bounded holomorphic functions in the unit disk is a Banach algebra under the supremum norm. The unit disk is homeomorphically embedded (in an obvious way) in the Gelfand space (maximal ideal space) of H^∞. Is the unit disk dense in the Gelfand space of H^∞? Question 1 was raised and answered by K. deLeeuw and W. Rudin **[9]**. They showed that the extreme points of the unit ball of H^1 are the outer functions of unit norm. Question 2 was answered independently by Beurling (unpublished) and Rudin **[10]**. The description of the closed ideals in the disk algebra involves the inner-outer factorization and is related to Beurling's H^2 invariant subspace theorem. The affirmative answer to Question 3 is the famous corona theorem of L. Carleson **[11]**—more on that later.

The classical results of one-dimensional function theory generalize in diverse and sometimes surprising ways. Extensions to one-dimensional domains other than the disk came early; already in their 1916 paper, F. and M. Riesz used their theorem to extend Fatou's results on nontangential boundary values to Jordan domains with rectifiable boundaries. The basic theory for a half-plane was worked out in the 1930s by E. Hille and J. D. Tamarkin **[12]** and others. For applications in operator theory, prediction theory and systems theory, one needs to extend portions of H^p theory, for example, the inner-outer factorization, to vector-valued and operator-valued functions. Such extensions, although often by no means routine, have been quite successful **[13–15]**. Attempts to extend the theory to several complex variables, on the other hand, have been more tentative, and much remains unknown. The one-dimensional theory has pointed to fruitful directions, but the existing one-dimensional techniques are often inadequate.

Large portions of the classical theory of Hardy spaces have been extended to the setting of abstract function algebras **[16]**, a development which has considerably enriched the classical theory. The same can be said, even more emphatically, in connection with extensions to Euclidean spaces **[17, 18]**. Here, the classical complex variable techniques are not available, of course; one must adopt a thoroughly real variable attitude. Not only have powerful new techniques been invented, but basic new results—new even to the one-dimensional theory—have been discovered.

The F. and M. Riesz theorem typifies this entire area of mathematics. The original proof relies upon certain constructions with holomorphic functions. The theorem has an alternative formulation: every H^1 function is the Poisson integral of its boundary function, and, unless the function is identically zero, the boundary function is nonzero almost everywhere. F. Riesz's 1923 paper contains a proof of this version, involving factorization, which is based on completely different ideas from the original proof. The first generalization of the theorem I know of, due to S. Bochner, appeared in 1944 **[19]**; it states that a measure on a torus whose Fourier coefficients are confined to an octant is absolutely continuous with respect to Lebesgue measure. The proof depends on one of the Hardy-Littlewood maximal theorems and specializes to a proof of the original theorem different from the two already mentioned. In 1958, H. Helson and D. Lowdenslager published an influential paper [**20**] in which they extended much of the classical H^p theory to the context of compact Abelian groups with ordered duals; their techniques are from real variables and elementary Hilbert space theory. In particular, their paper contains a version of the F. and M. Riesz theorem from which one can deduce Bochner's version. A new proof of the original F. and M. Riesz theorem results which is totally different from any of its predecessors—even more different from them than they are from each other. The new proof can be rephrased in a way that makes the Riesz theorem depend on elementary properties of unitary operators [**21**]. In 1963, F. Forelli [**22**] presented a proof of the Helson-Lowdenslager theorem which is close in spirit to the original proof of F. and M. Riesz (and to a later one of Helson [**23**]).

The Helson-Lowdenslager theory evolved during the 1960s into the theory of function algebras. During the process of evolution many abstract versions of the F. and M. Riesz theorem surfaced and played leading roles. The ultimate version, in this direction, is due to I. Glicksberg [**24**]; its proof is an adaptation of Forelli's. In another direction, deLeeuw and Glicksberg extended the Helson-Lowdenslager version to a more general group-theoretic context [**25**], and Forelli extended it to dynamical systems [**26**].

There are yet other extensions. For instance, in their 1916 paper, the Riesz brothers drew motivation from a geometric consequence of their theorem: on a rectifiable Jordan curve in the plane, harmonic measure (evaluated at an interior point) and arc length measure are mutually absolutely continuous. That result has been extended to bounded Lipschitz domains in Euclidean spaces by B. Dahlberg [**27**].

The discussion above affords a too narrow but, I hope, a suggestive glimpse of the vast area I am calling one-dimensional function theory. The area is of great interest to many mathematicians whose primary focus is elsewhere. Others are attracted to it by its intrinsic beauty, a quality that the book under review succeeds in conveying.

The book is concerned with the theory in the unit disk and the upper half-plane. The basic earlier results, from Fatou to Beurling, are developed in the first three chapters, after which the material comes predominantly from the last twenty years. During that time the two deepest and farthest reaching results have been Carleson's corona theorem and C. Fefferman's duality theorem.

The corona theorem, that the unit disk is dense in the Gelfand space of H^∞, has a more concrete formulation: if $f_1,\ldots,f_n$ are functions in H^∞ such that $|f_1(z)| + \cdots + |f_n(z)|$ is bounded away from 0 for $|z| < 1$, then $f_1,\ldots,f_n$ generate H^∞ as an ideal (that is, there are $g_1,\ldots,g_n$ in H^∞ such that $f_1g_1 + \cdots + f_ng_n = 1$). This formulation suggests certain interpolation problems, notably, the problem of characterizing the so-called interpolating sequences for H^∞ (sequences (z_n) in the unit disk such that the map $f \to (f(z_n))$ from H^∞ to l^∞ is onto). The latter problem was solved by Carleson in 1958 [**28**]. The heart of Carleson's proof of the corona theorem is a complicated construction which produces, for a given function f in H^∞, a system of curves surrounding the set where f is small and possessing certain additional properties. The curves enabled Carleson to solve an interpolation problem D. J. Newman had previously shown implies the corona theorem. L. Hörmander **[29]** subsequently introduced a less complicated approach involving the $\bar{\partial}$-equation which evades Newman's argument. It still needs Carleson's construction, however. Carleson's construction has found important applications besides the corona theorem. Our author was the first person after Carleson to use it in a significant way.

Fefferman's duality theorem **[30]** identifies the dual of the Hardy space H^1 as BMO, the space of functions of bounded mean oscillation. (The actual statement of the theorem is more precise, of course.) The space H^1 here is not the one introduced earlier in this review but a real variables version which can be defined in Euclidean spaces. The theorem emerged as part of the program to extend the classical Hardy space theory to Euclidean spaces and was not foreshadowed by any one-dimensional development. Its impact was sudden and pervasive.

The theorems of Carleson and Fefferman are linked through the notion of a Carleson measure. A (one-dimensional) Carleson measure is a positive measure m on the unit disk (or upper half-plane) with the property that the Poisson integral defines a bounded map from L^p of the unit circle (or real line) to $L^p(m)$, $1 < p < \infty$. Carleson gave a geometric characterization of such measures. The chief difficulty in Carleson's corona construction derives from the requirement that arc length measure on the constructed curves be a Carleson measure. By achieving that property, Carleson was able to use the curves to make certain estimates needed to solve Newman's interpolation problem. Hörmander's approach to the corona theorem is based on a connection between Carleson measures and the existence of bounded solutions of $\bar{\partial}$-equations. In Fefferman's theorem, Carleson measures come up in a characterization of BMO functions. Insight into the relation between Carleson's theorem and Fefferman's enabled T. Wolff in 1979 to devise a proof of Carleson's theorem (actually, of a refinement of it) which avoids the Carleson construction.

Garnett's book contains thorough and very informative chapters on BMO and on the corona theorem. The former chapter contains not only Fefferman's theorem and the needed preliminaries but many of the theorem's ramifications as well. The latter chapter presents both Wolff's proof and, in very understandable fashion, the original Carleson construction. The book contains much else besides—interpolation problems, a little on abstract function algebras, a great deal on concrete function algebras, properties of the conjugation operator, and approximation problems are some of the other ingredients. I refer the reader to my review in *Mathematical Reviews* for a more systematic outline of the contents.

The book of Garnett is not an exhaustive account of one-dimensional function theory; no reasonably sized book could be. Its most noticeable lack is in the direction of operator theory. Within its scope, however, it pursues the subject to considerable depth. Real variable methods, emphasizing the role of maximal functions, are employed whenever possible, so the book should be especially useful to someone intending to pursue the theory in higher dimensions. The style, while perhaps a little mechanical and curt, is nevertheless clear and precise. The author has exercised great care in presenting the material and has restructured many a proof into more understandable form. I can highly recommend the book as an excellent source for many of the deeper advances in one-dimensional function theory during the past two decades. (The beginner in the subject, though, will probably appreciate a more leisurely introduction; for that I recommend the recent book of P. Koosis **[31]**.)

To repeat myself, many pursue one-dimensional function theory for its own sake while many are attracted there initially due to the demands of some other specialty. Both parties will find much of value in Garnett's book.

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Winning ways for your mathematical plays, by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, Academic Press, London, 1982, vol. 1, *Games in general*, vol. 2, *Games in particular*, xxxii + 861 pp., $22.50 P/B, $64.50 H/B.

Winning ways is a masterpiece. We should have been disappointed were it anything less. Fifteen years in the preparation, and representing the collaboration of three mathematicians of extraordinary talent, the result is the most compelling and comprehensive treatment of mathematical games to appear in this century.

First, an enumeration of some of the things which this book is *not*. It has an empty intersection with "the Theory of Games" in the sense of von Neumann and Morgenstern [**6**]. More generally, it avoids discussion of "games" in which randomizing elements (the roll of dice, the shuffling of cards, the spinning of discs, or other methods of selecting a "move" in a stochastic fashion) play any role. This leaves full information, "deterministic" games such as chess, checkers (draughts), and Go, in which two players move alternately. However, these three examples of games actually played by adult humans are far too complicated to be analyzed in *Winning ways*.

Winning ways is published in two volumes. The first volume (WWI) is subtitled *Games in general*, while the second (WWII) is subtitled *Games in particular*. Each volume in turn consists of two parts. The four parts are associated successively with Spades, Hearts, Clubs, and Diamonds, but this has no underlying significance, and is for identification purposes only.

"Spade-work" (the first eight chapters) develops the generalized theory for analyzing and evaluating Nim-like games. The published analysis of Nim itself [**1**] goes back to C. L. Bouton in 1902. In 1939, P. M. Grundy [**5**] published a method for the recursive evaluation of positions in a relatively large class of Nim-like games, and this evaluation function for such a game became known