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Numerical analysis of variational inequalities, by R. Glowinski, J. L. Lions and R. Trémolières, Studies in Mathematics and its Applications, vol. 8, North-Holland, Amsterdam, 1981, xxx + 778 pp., $109.75.

Consider the problem of minimizing a real-valued function f over a space V. If u attains the minimum and f is differentiable at u, then $f'[u] = 0$. On the other hand, if K is a convex subset of V and u is optimal for the problem

$$\text{minimize } \{f(v): v \in K\}, \tag{1}$$

then an inequality holds,

$$f'[u](v-u) \geqslant 0 \qquad \forall v \in K. \tag{2}$$

Loosely speaking, (2) says that f increases when we move from u into K. The book by Glowinski, Lions, and Trémolières studies numerical aspects of (1) and (2) for a broad class of physical problems.

The obstacle problem illustrates the type of inequality included in their analysis: Given an open set $\Omega \subset R^2$ and functions $f \in \mathcal{L}^2(\Omega)$ and $\psi \in \mathcal{H}^2(\Omega)$,

$$\text{minimize } \int_\Omega \left\{ \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + 2fv \right\} dx\, dy$$

subject to $v \in \mathcal{H}^1(\Omega)$, $v = 0$ on $\partial\Omega$, $v \geqslant \psi$ almost everywhere in Ω. Here $\mathcal{L}^2(\Omega)$ is the space of real-valued functions that are square integrable on Ω, and $\mathcal{H}^k(\Omega) \subset \mathcal{L}^2(\Omega)$ is the Sobolev subspace consisting of functions whose derivatives through order k lie in $\mathcal{L}^2(\Omega)$. The function ψ is the obstacle. In this context, it can be shown [3] that the inequality (2) is equivalent to the relations

$$\left.\begin{array}{l} u \geqslant \psi \\ f \geqslant \Delta u \\ (f - \Delta u)(\psi - u) = 0 \end{array}\right\} \quad \text{almost everywhere in } \Omega.$$

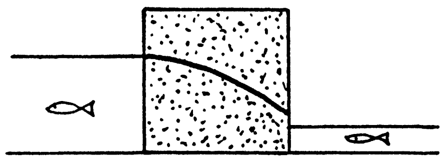
These relations tell us that $u = \psi$ on part of Ω while $u > \psi$ and $\Delta u = f$ on the complement. The curve that forms the boundary of $\{x \in R^2: u(x) > \psi(x)\}$ is often called the *contact set*.

Many physical problems have the form (1) or (2), and the book by Duvaut and Lions [6] is a good reference on this subject. For example, in plasticity theory, the stress is constrained to lie inside a yield surface. The stress potential for an elastic-perfectly plastic cylindrical bar undergoing torsion is the solution to the problem

$$\text{minimize } \int_\Omega \left\{ \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + 2fv \right\} dx\, dy$$

subject to $v \in \mathcal{H}^1(\Omega)$, $v = 0$ on $\partial\Omega$, $(\partial v/\partial x)^2 + (\partial v/\partial y)^2 \leqslant 1$ almost everywhere in Ω where Ω is the bar's cross-section, and the constraint $|\nabla v|^2 \leqslant 1$ is the von Mises yield criterion. Other problems studied by the authors involve the diffusion of fluids through semipermeable membranes, the displacement of a solid subject to boundary friction, the flow of a Bingham fluid in a cylindrical duct, impulsive stochastic control, and transonic potential flow.

A discovery by Baiocchi [1] in 1971 added to the excitement surrounding variational inequalities in the 1970s. He found the free boundary problem associated with seepage through an earth dam is equivalent to an inequality.



Since water seeps through the earth, we observe in the dam's cross-section an air-water interface called the *free boundary*. Using a clever transformation, Baiocchi shows that this "dam" problem is equivalent to the obstacle problem, and the free boundary is the contact set. Other connections between free boundary problems and variational inequalities are contained in [**2** and **5**] and the papers cited in their bibliographies.

Glowinski, Lions, and Trémolières study numerical questions associated with variational inequalities: If (1) or (2) is approximated by a finite element or a finite difference scheme and u^h denotes the discrete approximation, does u^h converge to u as the mesh parameter h tends to zero? How fast does u^h converge to u? How is the discrete analogue of (1) or (2) solved? Chapter 1 presents various formulations of the model-variational problems, and develops a general framework for proving the convergence of u^h to u. Chapter 2 summarizes mathematical programming algorithms, while the remaining chapters apply the convergence results and the numerical algorithms to the model problems. Many numerical computations are presented, and the relative merits of algorithms and discretizations are evaluated.

This book helps consolidate and organize the enormous literature on variational inequalities. The bibliographic citations are grouped below into time periods.

Period	Papers Cited
1847–1949	8
1950–1959	26
1960–1969	117
1970–1980	281

During the 1970s, there was an explosion in research on variational inequalities —a relevant book or paper appeared at least every 15 days on the average. Moreover, in 1977 Cryer [**4**] compiled a *selective* bibliography of 3300 references concerning free boundary problems. In this setting of daily discoveries, Glowinski, Lions, and Trémolières developed their book. The French version [**7**] appeared in 1976. The subsequent English translation, the subject of this review, is supplemented by 236 pages of appendices that bring the book up-to-date, and provide a more systematic treatment of some topics from the French version. In summary, the authors have prepared a valuable reference for mathematicians and engineers.

REFERENCES

1. C. Baiocchi, *Sur un problème à frontière libre traduisant le filtrage de liquides à travers des milieux poreux*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), 1215–1217.

2. C. Baiocchi, F. Brezzi and V. Comincioli, *Free boundary problems in fluid flow through porous media*, Second International Symposium on Finite Element Methods in Flow Problems (Santa Margharita, 1976), pp. 407–420.

3. H. Brézis and G. Stampacchia, *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France **96** (1968), 153–180.

4. C. W. Cryer, *A bibliography of free boundary problems*, Mathematics Research Center Report #1793, 1977.

5. C. W. Cryer and H. Fetter, *The numerical solution of axisymmetric free boundary porous well problems using variational inequalities*, Mathematics Research Center Report #1761, 1977.

6. G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

7. R. Glowinski, J. L. Lions, and R. Trémolières, *Analyse numerique des inéquations variationnelles*, Dunod, Paris, 1976.

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Bounded analytic functions, by John B. Garnett, Academic Press, New York, 1981, xvi + 468 pp., $59.00.

The book under review belongs to an area which, for want of a better term, I shall call one-dimensional function theory. "Function theory" should be interpreted here, not in the old sense of the theory of functions of a complex variable, but rather in a broader sense encompassing both the analysis of functions, holomorphic or not, and the analysis of spaces of functions. The settings for one-dimensional function theory are primarily the unit disk and the upper half of the complex plane together with their boundaries, the unit circle and the real line, respectively.

One-dimensional function theory is not a branch of mathematics in the way that, say, operator theory and low-dimensional topology are. Perhaps it does not even deserve a name of its own. The operator theorist seeks to understand the structure of operators, the low-dimensional topologist to understand the structure of three-dimensional and four-dimensional manifolds. The practitioner of one-dimensional function theory is aware of no comparable ultimate goal. This in part reflects the status of one-dimensional function theory as a handmaiden of several other, more coherent, disciplines—operator theory, theory of Banach spaces and topological vector spaces, prediction theory, systems theory, theory of commutative Banach algebras—which it provides