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ON THE SCHROEDINGER CONNECTION

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A new and more direct approach to the connection of wave amplitudes across turning points and singular points of wave- and oscillator-equations has been found. It emphasizes and extends the view [1] that the connection formulae are an asymptotic expression of the branch structure of the singular point and reveals an unexpected two-variable structure even close to such points. It also extends turning-point theory to new classes of *irregular* points of differential equations

$$\epsilon^2 d^2 w/dz^2 + p(z)w(z) = 0 \tag{1}$$

with constant ϵ and analytic $p(z)$ that are physical Schroedinger equations in the sense that the concept of wavelength (or period) can be defined.

A natural (Liouville-Green) variable x measured in units of local wavelength is then also definable. Limit points of singular points of $p(z)$ will be excluded, as will singular points artificially introduced to represent radiation conditions. Any turning- or singular point of $p(z)$ must then correspond to a definite x, and if those points be identified with $z = 0$ and $x = 0$, respectively, then

$$x = \frac{i}{\epsilon}\int_0^z [p(t)]^{1/2} dt \tag{2}$$

must exist, at least as a multivalued function, on a neighborhood of zero.

For a clear theory, this hypothesis should be rephrased in terms of the natural variable: an analytic branch $r(x)$ of $p^{1/4}$ is defined on a Riemann surface element D about $x = 0$ which includes $-\pi < \arg x < 2\pi$ (i.e., three Stokes sectors, in turning-point terminology) so that $i dz/dx = \epsilon/r^2$ is integrable at $x = 0$.

In the natural variable, with $w(z) = y(x)$, (1) takes the form

$$y'' + 2r^{-1}r'y' = y, \qquad r'/r = (\epsilon/2ip)d(p^{1/2})/dz, \tag{3}$$

and wave modulation is therefore controlled by r'/r; since $p = p(z)$, also ϵx depends only on z, by (2), and xr'/r depends on x and ϵ only through the product ϵx, by (3). Turning points and singular points of (1) are all singular points of (3), and when they do not lie on the real axes of z or x, physics places no further, general restriction on their nastiness. For the results here reported, the following, secondary hypothesis has been found sufficient: a limit

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of xr'/r can be identified, i.e.,

$$xr'/r \to \gamma \in \mathbf{C} \quad \text{as } \epsilon x \to 0,$$

uniformly in the Riemann surface element on which r has been defined. Equivalently,

$$p^{1/4} = r(x) = (\epsilon x)^{\gamma} \rho(\epsilon x)$$

with a function $\rho(\xi)$ analytic on its domain Δ of definition and "*mild*" in the sense

$$(4) \qquad (\xi/\rho) d\rho/d\xi = \phi(\xi) \to 0 \quad \text{as } \xi \to 0, \text{ uniformly in } \Delta.$$

As a consequence, ρ varies less than any nonzero real power,

$$\forall \nu > 0, \quad |\xi^{\nu} \rho^{\pm 1}| \to 0 \quad \text{as } \xi \to 0,$$

and γ represents the "nearest power" of x in $r(x) = p^{1/4}$. The integrability of ϵ/r^2 implies $\operatorname{Re}\gamma \leq \frac{1}{2}$.

The class of singular points thus defined includes all turning points of second-order equations in the literature [**2**]; it extends even the class of [**3**]. Note the arbitrary multivaluedness of $r(x)$ and $p(z)$. Like the definition of $r(x)$ and $p(z)$, that of the differential equation (1) is purely *local*, described by

$$z^{-1} \int_0^z [p(t)/p(z)]^{1/2} dt \to 1 - 2\gamma \in \mathbf{C} \quad \text{as } z \to 0.$$

For $\epsilon = 0$, also $\phi(\epsilon x) = 0$ in (4), and the singular point is regular; the irregularity function ϕ therefore establishes a diffeomorphism between irregular and regular points. The originally superfluous constant ϵ in (1) reveals itself as a homotopy parameter indicating a general avenue of approach to irregular points from regular ones.

The branch structure of a regular point can be characterized by Frobenius' fundamental system [**1**, p. 149] $f_s(x)$, $x^{1-2\gamma} f_m(x)$ with (usually) entire f_s, f_m (and $f_s(0) = f_m(0) = 1$). Irregular points have a similar f.s. $y_s(x), y_m(x)$ with distinct branch points [**4**]:

THEOREM 1. *If $|\epsilon x|$ is not too large,* (3) *has a solution $y_m(x) = z(x)\hat{y}(x)$ analytic on D with*

$$z(x) = x^{1-2\gamma} \varsigma(\epsilon x), \qquad \hat{y}(x) = 1 + \sum_1^{\infty} \alpha_n(\epsilon x)(x/2)^{2n}$$

with mild (in the sense of (4)*), but generally multivalued ς and α_n; the series has a majorant power series in x of infinite convergence radius.*

THEOREM 2. *For noninteger $\frac{1}{2} - \operatorname{Re}\gamma$ and small enough $|\epsilon x|$,* (3) *has a solution*

$$y_s(x) = 1 + \sum_1^{\infty} \beta_n(\epsilon x)(x/2)^{2n}$$

analytic on D with mild and bounded, but generally multivalued, β_n; and the convergence radius is again infinite.

Observe the two-variable structure in terms of x and ϵx and that the merely local definition of (1) has led to a solution representation of global nature in x, even if local in ϵx,—a mathematical key to wave modulation and asymptotic connection. As $\epsilon x \to 0$, $y_s(x)$ and $\hat{y}(x)$ approach evenness, which suggests a characterization [4] of the departure of $y_s, \hat{y}$ from the entirety of their counterparts f_s, f_m (which are even for (3)):

THEOREM 3. *For x and $xe^{-\pi i}$ in D and not too large $|\epsilon x|$,*

$$|\hat{y}(x) - \hat{y}(xe^{-\pi i})| \leq \delta_m(|\epsilon x|)\Gamma(m)|x/2|^{2-m} I_m(|x|)$$

and $\delta_m(|\epsilon x|) \to 0$ as $|\epsilon x| \to 0$. For noninteger $\frac{1}{2} - \operatorname{Re}\gamma$ and small enough $|\epsilon x|$, also

$$|y_s(x) - y_s(xe^{-\pi i})| \leq \delta_s(|\epsilon x|)C(\gamma)|x/2|^{2-s} I_s(|x|)$$

and $\delta_s(|\epsilon x|) \to 0$ as $|\epsilon x| \to 0$.

Here $m = 3/2 - \operatorname{Re}\gamma - \operatorname{lub}|\phi(\epsilon x) + (\epsilon x/\varsigma)d\varsigma/d(\epsilon x)| > 0$, $s = \frac{1}{2} + \operatorname{Re}\gamma - \operatorname{lub}|\phi(\epsilon x)|$ and I denotes the modified Bessel function. As $|\epsilon x| \to 0$, y_s and $\hat{y}$ therefore tend to even functions of x uniformly on compacts; for fixed $|\epsilon x|$, their oddness can grow at most exponentially with $|x|$.

Integer values of $\frac{1}{2} - \operatorname{Re}\gamma$ correspond to the Frobenius exceptions where f_s has a logarithmic branch point [**1**, p. 150], and y_s can then be obtained [**4**] from a different representation of this solution, but loses the symmetry bound of Theorem 3.

Far from a singular point, the solutions of genuine Schroedinger equations are wave-like. More precisely, $r(x)y(x) = W(x)$ satisfies $W'' = (1 + r''/r)W$ with $r''/r = x^{-2}[\gamma(\gamma-1) + \phi\{2\gamma - 1 + \phi + \epsilon x\phi'/\phi\}]$ absolutely integrable along paths in D bounded from $x = 0$ so that [**1**, p. 222] a "WKB" solution pair

$$W_+(x) = a(x)e^x, \qquad W_-(x) = b(x)e^{-x}$$

exists with a, b analytic on D and bounded for large $|x|$ (provided $|\epsilon x|$ is slightly restricted so that ϕ and $\xi\phi'$ are bounded). The decay of $|r''/r|$ at large $|x|$ also assures [**1**, pp. 223, 224] limits of a, b as $|x| \to \infty$ with $(\arg x)/\pi$ an integer, which are *wave-amplitudes* of (1).

Any solution must be a linear combination of W_+, W_-, i.e.,

$$r(x)y_m(x) = \tilde{a}_m(x)e^x + \tilde{b}_m(x)e^{-x}, \tag{5}$$

and similarly with subscript s, with similarly bounded $\tilde{a}_m, \ldots, \tilde{b}_s$, some of which must be multivalued like ry_m. Connecting wave-amplitudes of (1) therefore means [**1**, p. 481] finding "circuit relations", i.e., the difference between the respective limits of $\tilde{a}_m$, etc., as $|x| \to \infty$ with $\arg x = 0$ and as $|x| \to \infty$ with $\arg x = 2\pi$. If such a limit of a function $f(x)$ as $|x| \to \infty$ with $\arg x = \sigma$ is abbreviated as $f(\infty e^{i\sigma})$ [and if $\sigma = 0$, then by $f(\infty)$], a typical connection question reads briefly $\tilde{a}_m(\infty \exp 2\pi i) - \tilde{a}_m(\infty) = ?$

However, $\tilde{a}_m, \ldots$ are normalized via y_m, y_s, which introduces an ϵ-dependence, and since $|x|$ is bounded on D for fixed $\epsilon \neq 0$, the connection question can be asked only in the limit $\epsilon \to 0$. Scrutiny of the normalization [**5**] shows that

$$\tilde{a}_m/(\rho\varsigma) = a_m, \qquad \tilde{b}_m/(\rho\varsigma) = b_m, \qquad \tilde{a}_s/\rho = a_s, \qquad \tilde{b}_s/\rho = b_s,$$

rather than $\tilde{a}_m, \ldots$, are certain to tend to limits as $\epsilon \to 0$ and $|x| \to \infty$. Directly meaningful connection questions should therefore be phrased like $a_m(\infty \exp 2\pi i) - a_m(\infty) = ?$

Now, if $\exp(-\pi i)$ is abbreviated by j and if x and jx are in D, then (5) at x and jx implies the further identity

$$(6) \qquad \begin{aligned}[y(x) - y(jx)]x^{1-\gamma}e^{-|x|} = {} & [a_m(x) - j^{\gamma-1}b_m(jx)]e^{x-|x|} \\ & + [b_m(x) - j^{\gamma-1}a_m(jx)]e^{-x-|x|}\end{aligned}$$

on D. Remarkably, Theorem 3 permits us to let $|x| \to \infty$ while $|\epsilon x| \to 0$ so that the left-hand side of (6) still tends to zero! E.g., $|x| = |\log \delta_m(|\epsilon x|)|$ serves. The choices $\arg x = 0, \pi, 2\pi$ then imply the connection answers

$$\begin{pmatrix} a_m(\infty) \\ b_m(\infty e^{\pi i}) \\ a_m(\infty e^{2\pi i}) \end{pmatrix} = e^{(1-\gamma)\pi i} \begin{pmatrix} b_m(\infty e^{-\pi i}) \\ a_m(\infty) \\ b_m(\infty e^{\pi i}) \end{pmatrix}$$

whence also

$$(7) \qquad \begin{aligned} a_m(\infty e^{2\pi i}) - a_m(\infty) &= 2i \sin(\gamma\pi)\, b_m(\infty e^{\pi i}), \\ b_m(\infty e^{\pi i}) - b_m(\infty e^{-\pi i}) &= 2i \sin(\gamma\pi)\, a_m(\infty). \end{aligned}$$

For y_s, (6) holds with y_s, s and γ in the respective places of y, m and $1-\gamma$, and if $\frac{1}{2} - \operatorname{Re}\gamma$ is not an integer, Theorem 3 leads to (7) also with subscript s. Hence, (7) holds for any solution $y(x) = w(z)$ of (1), with interpretation appropriate to the normalization of that solution. Analytic continuation in γ extends [5] the connection formulae (7) to all γ with $\operatorname{Re}\gamma < \frac{1}{2}$.

REFERENCES

1. F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.
2. R. E. Langer, *On the asymptotic solutions of differential equations, with an application to Bessel functions of large complex order*, Trans. Amer. Math. Soc. **34** (1932), 447–464.
3. J. F. Painter and R. E. Meyer, *Connection at close quarters to generalized turning points*, SIAM J. Math. Anal. **13** (1982), 541–554.
4. ——, *Irregular points of modulation*, MRC Technical Summary Report #2264, 1981; Adv. Appl. Math. (to appear).
5. ——, *Connection for wave modulation*, MRC Technical Summary Report #2265, 1981; SIAM J. Math. Anal. (to appear).

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