Propriétés spectrales des algèbres de Banach, by Bernard Aupetit, Lecture Notes in Math., vol. 735, Springer-Verlag, Berlin and New York, 1979, xii + 192 pp., $13.80.

0. Introduction. From the beginning of Banach algebra theory in the fundamental papers of Gelfand and his collaborators in the 1940s, particularly the classic paper [6], the spectrum has played a key role both in the general theory and in its many applications. Recall that if a belongs to the complex Banach algebra A with identity, then the *spectrum*, $\mathrm{Sp}(a)$ or $\mathrm{Sp}_A(a)$, of a is the set of complex numbers λ for which $\lambda - a$ is not invertible in A. Recall also that $\mathrm{Sp}(a)$ is a compact nonvoid subset of complex numbers [3, 15]. To avoid technicalities in this review, we will not discuss real Banach algebras or algebras without identity, though Aupetit and the standard treatises [3, 15] sometimes do.

The spectrum is an important and natural concept in the standard examples of Banach algebras that occur in applications. For operators on a Banach space the spectrum is the usual operator spectrum, and the essential spectrum or Fredhölm spectrum is just the spectrum in the Calkin algebra. For commutative group algebras like $L^1(\mathbf{R})$ with an identity adjoined, the spectrum of a function is just the range of its Fourier transform union $\{0\}$.

For $C(X)$, the algebra of continuous functions on the compact Hausdorff space X, the spectrum of a function is its range. For an arbitrary commutative Banach algebra A the *Gelfand transformation* [6] is a continuous algebra homomorphism $a \to \hat{a}$ from A to an appropriate $C(X)$ and, moreover, $\mathrm{Sp}_A(a) = \hat{a}(X)$. Thus the Gelfand transformation shows that the behavior of the spectrum is particularly simple in commutative Banach algebras. One focus of recent research is to determine to what extent simple spectral behavior entails commutativity.

The two standard treatises on Banach algebras [15, 3] were published in 1960 and 1973, respectively, and thus do not describe the enormous number of interesting results discovered in the last decade. Aupetit's book is a very comprehensive report on those recent results which are related to spectral theory, many of which are due to Aupetit himself. (For excellent reports on the most important other recent results, see [2, 5, 17].) From now on we will usually refer to Aupetit's book as [AUP].

Many of the recent advances in spectral theory come from the exploitation, largely by Aupetit, but also by Jaroslav Zemánek and his collaborators, of the subharmonicity of various functions of the spectrum. Explicitly, if we let $\rho(a) = \max(\mathrm{Sp}(a))$ be the *spectral radius* of a and let $\delta(a)$ be the diameter of $\mathrm{Sp}(a)$, and if $f(\lambda)$ is any analytic function from a domain of complex numbers to a Banach algebra, then $\rho(f(\lambda))$, $\log \rho(f(\lambda))$, $\delta(f(\lambda))$, and $\log \delta(f(\lambda))$ are all subharmonic [AUP, pp. 9–11]. In applications, $f(\lambda)$ is usually an explicit

function like $a + \lambda(x - a)$, $(\lambda - a)^{-1}$, or $e^{\lambda b} a e^{-\lambda b}$, and three kinds of properties of subharmonic functions play a key role.

(1) If $h(\lambda)$ is a subharmonic function then $\lim \sup_{\lambda \to \lambda_0} h(\lambda) = h(\lambda_0)$. Moreover, this remains true if we restrict λ to a set E which is "nonthin" at λ_0, in particular if λ_0 belongs to the closure of a nontrivial component of E [AUP, pp. 170–171].

(2) A subharmonic function satisfies certain growth conditions like the maximum modulus property and Liouville's theorem.

(3) If a subharmonic function like $\log \rho(f(\lambda))$ is $-\infty$ "often enough" then it is identically $-\infty$. Here "often enough" means on a set with positive outer logarithmic capacity. A set has positive outer capacity if, for example, it has nonvoid interior, or positive Lebesgue or Hausdorff measure, or is not totally disconnected [AUP, pp. 172–173].

As a first easy illustration of the application of subharmonic functions, Aupetit shows [AUP, Corollary 2, p. 11] that if the set of *quasinilpotent* elements, that is elements with $\{0\}$ spectrum, has an interior point a, then all elements are quasinilpotent (because $\rho(a + \lambda(x - a)) \equiv 0$ for each x).

In the next three sections, we will describe some typical results about spectral continuity, commutativity, and spectral finiteness, which are the three major areas of spectral theory and its applications discussed by Aupetit.

1. Spectral continuity. The earliest results on spectral continuity are due to Newburgh [14] (see [AUP, pp. 6–9]). Newburgh shows that the spectrum is *upper semicontinuous* in the sense that given a in A and $\varepsilon > 0$, there is a $\delta > 0$ for which $\| x - a \| < \delta$ implies that $\text{dist}(\lambda, \text{Sp}(a)) < \varepsilon$ for all λ in $\text{Sp}(x)$. When $\text{Sp}(a)$ is totally disconnected he also proves the reverse inequality $\text{dist}(\mu, \text{Sp}(x)) < \varepsilon$ for all μ in $\text{Sp}(a)$; so that in that case the *Hausdorff distance* $\Delta(\text{Sp}(a), \text{Sp}(x)) < \varepsilon$; and hence $\text{Sp}(x)$ is continuous at a in the Hausdorff metric. Conway and Morrel improve this result slightly [4, Proposition 2.4, p. 57] as a step in their determination of necessary and sufficient conditions for spectral continuity at a point in the algebra of operators on Hilbert space (see also [13]).

Newburgh also shows that if we restrict $\text{Sp}(x)$ to the set of elements commuting with a, then this restriction is continuous at a (cf. [15, Theorem 1.6.17, p. 36]). It is easy to see from the Gelfand transformation that on a commutative Banach algebra the spectrum is uniformly continuous, but it is still not clear on which Banach algebras the spectrum is continuous. On the one hand, if B is commutative then the spectrum is continuous on a class of algebras which include all Banach subalgebras of the algebra of $n \times n$ matrices with entries in B [AUP, pp. 138–140]. But on the other hand there are many examples of spectacularly discontinuous spectra [AUP, pp. 34–41].

Since spectral continuity is so rare in a noncommutative Banach algebra, several people have tried to find conditions on a subset N which guarantee that $\text{Sp}_A(x)$ is continuous when restricted to N. Again the first such result is due to Newburgh, who finds hypotheses on N sufficiently weak [14, p. 169] so that they are satisfied when N is the set of normal elements in a C^*-algebra [14, Corollary 2, p. 170]. In the mid 1970s several people extended Newburgh's

result to prove uniform continuity. Aupetit weakened the hypothesis somewhat more in order, for instance, to prove uniform continuity of the spectrum on the normal elements of a symmetric star algebra [AUP, pp. 142–143].

All the above continuity results use only elementary methods which have been available for decades. Using subharmonic functions Aupetit proves what he calls an "almost-continuity" theorem [AUP, Theorem 1, p. 32] whose major consequence is

THEOREM [AUP, COROLLARY 1, p. 33]. *Suppose that* $f(\lambda)$ *is an analytic function from a domain D of complex numbers to a Banach algebra A and that* $E \subseteq D$ *is "nonthin" at* λ_0 *in* $\bar{E} \cap D$. *If* Sp$(f(\lambda))$ *does not disconnect the plane, then there is a sequence* $\{\lambda_n\}$ *in E with limit* λ_0 *for which* Sp$(f(\lambda_n))$ *approaches* Sp $f(\lambda_0)$ *in the Hausdorff metric.*

As an application (cf. [AUP, p. 34]), suppose that Sp$(a + \lambda b)$ is real for $0 < \lambda \leqslant 1$; then Sp(a) is real.

2. Commutativity. The earliest commutativity results (see [AUP, pp. 42–47]), beginning with a fundamental 1967 paper of Le Page [12], mostly give norm conditions which imply commutativity. For instance one of Le Page's results [12, Proposition 2], [AUP, Theorem 1, p. 42] is that if there is a $k > 0$ for which $\| ab \| \leqslant k \| ba \|$ for all a and b in A, then A is commutative. In the mid 1970s, results which characterize commutativity in terms of spectral properties appear simultaneously in the works of Aupetit and of Zemánek and his collaborators. Since the spectrum of a in A is the same as the spectrum of its coset in A/Rad A [AUP, Lemma 2, p. 2] such results can only describe commutativity modulo the Jacobson radical of A. Of course, many of the Banach algebras that arise in applications, like operator algebras or group algebras, have $\{0\}$ radical; so that one actually characterizes commutativity in these cases. In his book Aupetit gives 13 properties equivalent to commutativity modulo the radical [AUP, Theorem 2, pp. 48–49]. From this list we have chosen three, which are also given by Zemánek [20, Theorem 2.1, p. 10].

THEOREM. *Each of the following conditions on A is equivalent to* A/Rad A *being commutative*:
(1) $\rho(x + y) \leqslant c(\rho(x) + \rho(y))$ *for some* $c > 0$.
(2) $\rho(xy) \leqslant m\rho(x)\rho(y)$ *for some* $m > 0$.
(3) $\rho(x)$ *is uniformly continuous on A.*

In the most up-to-date versions of the proof of the above theorem, the main step is the following lemma [20, Theorem 1.2, p. 6], [AUP, p. 52].

LEMMA. *If* sup $\rho(a - u^{-1}au)$ *is finite, then a is in the center of A modulo its radical.*

The earliest proofs of the lemma use the fact that if $g(\lambda) = (e^{\lambda b}ae^{-\lambda b} - a)/\lambda$ then $\rho(g(\lambda))$ is subharmonic (see [AUP, p. 49]), in much the same way that Le Page's proofs [10] use the fact that $g(\lambda)$ is analytic. The proof of the lemma is thus reduced to a result of Le Page [10, Proposition 6, p. 237], [AUP, Corollary 8, p. 46], which is based on the Jacobson-Rickart-Yood density theorem. More

recently Zemánek gave a beautifully simple purely algebraic proof [20, p. 6], [AUP, p. 52] of the lemma based on Allan Sinclair's extension on the density theorem [16, Theorem 6.7, p. 36], [AUP, Corollary 1, p. 160]. For most of the other characterizations of commutativity in [AUP, Theorem 2, pp. 48–49], subharmonicity seems to be essential to the proof, because those characterizations only give conditions valid in a suitable open set in A instead of in all of A.

In many ways the lemma is more interesting than the theorem because it gives a way to find central elements of a noncommutative algebra [AUP, Corollary 10, p. 51]. For algebras like the Calkin algebra or the algebra of operators on a Banach space which have trivial centers, one obtains striking characterizations of the scalar elements [20, p. 8].

3. Finite spectra. A number of recent papers have given conditions which imply that a Banach algebra or module is finite dimensional (see [AUP, pp. 69–77, 88–91] and the references he cites). These results usually use Kaplansky's finite spectrum theorem [11, Lemma 7, p. 376], which states that a Banach algebra in which every element has finite spectrum is finite dimensional modulo its radical. The theorem is often used in tandem with the reviewer's result that a nil Banach algebra is nilpotent [7] [AUP, Lemma 1, p. 88]. Aupetit is able to generalize Kaplansky's theorem by assuming only that the algebra has an open set of elements with finite spectrum [AUP, Theorem 1, p. 70], or, for a star algebra, that the Hermitian elements have a relatively open set of elements with finite spectrum [AUP, Theorem 2, p. 71].

Aupetit's extensions of Kaplansky's theorem are easy consequences of the following result, which is far and away the deepest and most substantial application of subharmonic functions to Banach algebras. The reader who works through the proof of the theorem [AUP, Theorem 1, p. 66] and of the lemmas it invokes will obtain a real appreciation of the power of subharmonic methods in Banach algebras.

THEOREM. *Suppose that $f(\lambda)$ is an analytic function from a domain of complex numbers D to a Banach algebra. If the set of λ for which $\mathrm{Sp}(f(\lambda))$ is finite has nonzero outer capacity, then there is a positive integer n for which $\mathrm{Sp}(f(\lambda))$ has no more than n points for all λ in D and has exactly n points except on a set with no limit points in D.*

Aupetit obtains his generalizations of Kaplansky's finite spectrum by applying the theorem just stated to $f(\lambda) = a + \lambda(x - a)$ where a is fixed and chosen appropriately and x is arbitrary.

By applying the lemma stated in the previous section, Zemánek obtains several interesting characterizations of the radical of a Banach algebra [20, pp. 16–17], [AUP, pp. 22–23]. For instance, if $a + q$ is quasinilpotent for all quasinilpotent q, then a belongs to the radical. In a similar way Aupetit uses the theorem stated in this section to show that if $a + f$ has finite spectrum whenever f has finite spectrum, then a is algebraic modulo the radical. These results have striking applications to the algebra of operators on a Banach space [19, Corollary 2, p. 4], [AUP, Corollary 7, p. 99] and to the Calkin algebra in

Hilbert space. For instance if the Hilbert space operator T is not polynomially compact, then there is an operator U with finite essential spectrum for which $T + U$ has infinite essential spectrum.

4. Concluding remarks. In this review we have emphasized a few highlights of the recent developments in spectral theory in general Banach algebras. Aupetit also discusses interesting recent applications of subharmonic functions and of spectral theory to special classes of algebras like function algebras, group algebras, and C^*-algebras. Particularly elegant is his treatment of Banach algebras with involution [AUP, pp. 107–130]. The most important results he discusses were obtained in the late 1960s, in time to be discussed in [3]. But Aupetit's treatment is somewhat more general and up to date and is completely independent of the theory of numerical ranges in Banach algebras. Among the interesting topics on general Banach algebras which we have not discussed is Robin Harte's exponential spectrum [8], [AUP, pp. 5–6] which generalizes the Weyl spectrum in the Calkin algebra to arbitrary Banach algebras, and Aupetit's extension of the Gleason-Kahane-Zelasko theorem from scalar-valued functions to matrix-valued functions [AUP, Theorem 5, p. 29].

An unusual feature of the development Aupetit chronicles is how often what later turn out to be key results were forgotten. Newburgh's theorems about spectral continuity at elements with totally disconnected spectra and on the set of normal elements [14] are described without proof by Rickart [15, p. 37] but they nevertheless seem to have been virtually unknown in the 1960s and early 1970s. Kaplansky's finite spectrum theorem [11] was so far forgotten that it was discovered all over again by two very good mathematicians when it was needed in the 1970s. Vesentini's 1968 proof of the subharmonicity of the spectral radius [18] seemed too special to be included in Bonsall and Duncan [3], and only became known because of Aupetit's spectacular applications of it in the mid 1970s. Sinclair's density theorem, which he published for the first time in his lecture notes [16] and which was shortly thereafter applied by Zemánek [20], actually was in Sinclair's Ph.D. thesis nearly ten years earlier. Even celebrated theorems like Johnson's uniqueness of norm theorem [10], [AUP, Theorems 4 and 5, pp. 161–162] often wait years before they are really used in applications. It appears that, in Banach algebra theory at least, elegant results of great generality eventually find important applications, provided they are not forgotten.

Aupetit's book has both the strengths and weaknesses of a set of lecture notes as distinct from a polished treatise. It often repeats results from papers with little change and no references to similar results elsewhere in the book. But the book is very thorough and up to date. If it were being written today, it would surely discuss Aupetit's subharmonic function proof of Johnson's uniqueness of norm theorem in [2], and Aupetit's joint work with Zemánek on rates of continuity [1]; but it is hard to think of other results which appeared too late to be included.

Aupetit is very thorough in tracing the history of results and in assigning credit, but the book very much emphasizes his personal point of view. For an emphasis on algebraic arguments one should consult Zemánek's beautiful

survey [20]; and for another discussion of the basic theory of subharmonic functions in Banach algebras one can consult Istrățescu [9, Chapter 14], who discovered the log-subharmonicity of the spectral radius almost simultaneously with Vesentini.

Since Aupetit's book has an appendix which contains the results he uses about subharmonic functions, it is essentially self-contained for anyone who knows the most basic facts about spectral theory and ideal theory in Banach algebra. Even those results are sketched rapidly in an appendix and in the first few pages of Chapter 1, though the reader is referred to [3 or 15] for proofs of the results connected with the Jacobson radical, which Aupetit uses in the discussion of commutativity and finiteness of spectra. Nonetheless a reader interested in a specific algebra or class of algebras with {0} radical, such as the Calkin algebra or group algebras, could probably learn the results he is interested in with little need to ready anything not in Aupetit. Such a reader will find many new results which will be interesting and valuable even for such specific algebras.

REFERENCES

1. B. Aupetit and J. Zemánek, *Local behavior of the spectral radius in Banach algebras*, J. London Math. Soc. 23 (1981), 171–178.

2. John Bachar et al, editors, *Proceedings of the conference on automatic continuity and radical Banach algebras*, Lecture Notes in Math., Springer-Verlag, Berlin and New York (to appear).

3. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973.

4. J. B. Conway and B. B. Morrel, *Behaviour of the spectrum under small perturbations*, Proc. Roy. Irish Acad. Sect. A. 81 (1981), 55–63.

5. H. G. Dales, *Automatic continuity: a survey*, Bull. London Math. Soc. 10 (1978), 129–183.

6. I. M. Gelfand, *Normierte Ringe*, Mat. Sb. 9 (1941), 3–21. (German, Russian summary)

7. S. Grabiner, *The nilpotency of Banach nil algebras*, Proc. Amer. Math. Soc. 21 (1969), 510.

8. R. Harte, *The exponential spectrum in Banach algebras*, Proc. Amer. Math. Soc. 58 (1976), 114–118.

9. V. I. Istrățescu, *Introduction to linear operator theory*, Dekker, New York, 1981.

10. B. E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc. 73 (1967), 407–409.

11. I. Kaplansky, *Ring isomorphisms of Banach algebras*, Canad. J. Math. 6 (1954), 374–381.

12. C. Le Page, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris Sér. A-B 265 (1967), 235–237.

13. G. J. Murphy, *Continuity of the spectrum and spectral radius*, Proc. Amer. Math. Soc. 82 (1981), 619–621.

14. J. D. Newburgh, *The variation of spectra*, Duke Math. J. 18 (1951), 165–176.

15. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, N. J., 1960.

16. A. M. Sinclair, *Automatic continuity of linear operators*, London Math. Soc. Lecture Note Series, No. 21, Cambridge Univ. Press, Cambridge, New York, Melbourne, 1976.

17. _____, *Continuous semigroups in Banach algebras*, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge (to appear).

18. E. Vesentini, *On the subharmonicity of the spectral radius*, Boll. Un. Mat. Ital. 4 (1968), 427–429.

19. J. Zemánek, *A note on the radical of a Banach algebra*, Manuscripta Math. 20 (1977), 191–196.

20. _____, *Properties of the spectral radius in Banach algebras*, Inst. of Math. Polish Acad. of Sci. Preprint 136, 1978; Proc. Spectral Theory Semester, Banach Center Publ. 8, PWN, Warsaw (to appear).

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