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NEW DEFECT RELATIONS FOR MEROMORPHIC FUNCTIONS ON $\mathbf{C}^n$

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For meromorphic functions f and φ on the complex line $\mathbf{C}^1$, one considers the *counting function* $N(r, \varphi, f) = \int_1^r n(t, \varphi, f)t^{-1}\,dt$, where $n(t, \varphi, f)$ denotes the number of solutions of the equation $f = \varphi$ (counting multiplicities) on the disk $\{|z| \leq t\}$. If $T(r, \varphi) = o(T(r, f))$, one defines the *defect* $\delta(\varphi, f) = \liminf[1 - N(r, \varphi, f)/T(r, f)]$ and observes that $0 \leq \delta(\varphi, f) \leq 1$ as in the case where φ =constant. In 1929, R. Nevanlinna [**4**, p. 77] asked if the *defect relation*

$$(*) \qquad \sum_{j=1}^{q} \delta(\varphi_j, f) \leq 2$$

is valid for distinct meromorphic functions φ_j with $T(r, \varphi_j) = o(T(r, f))$. The case where the φ_j are constant is Nevanlinna's fundamental defect relation [**4**]. (If $q = 3$, then $(*)$ follows immediately from the Nevanlinna defect relation.) In 1939, J. Dufresnoy [**3**] showed that $\sum \delta(\varphi_j, f) \leq d+2$ if f is transcendental and the φ_j are distinct polynomials of degree$\leq d$. In 1964, C.-T. Chuang [**2**] gave a general Second Main Theorem which yields $(*)$ for the case where f is holomorphic (or more generally when $\delta(\infty, f) = 1$) and which generalizes the defect relation of Dufresnoy [**3**]. However, this question of Nevanlinna remains unanswered today even for polynomial φ_j, despite Nevanlinna's assertion [**4**, p. 77] that $(*)$ "follows easily" for this case. If f is a meromorphic function on $\mathbf{C}^n$, then a special case of a theorem of W. Stoll [**7**] (see also Vitter [**8**]) yields $(*)$ for constant φ_j as in the classical Nevanlinna theory. (In fact, the results of Chuang [**2**] generalize easily to $\mathbf{C}^n$.) In this note we announce a new defect relation of the form $(*)$ for meromorphic functions on $\mathbf{C}^n$, $n \geq 2$.

If f and φ are distinct meromorphic functions on $\mathbf{C}^n$, we let $D(\varphi, f)$ denote the divisor on $\mathbf{C}^n$ given by the solution (with multiplicities) to the equation $f = \varphi$. We write $N(r, \varphi, f) = N(r, D(\varphi, f))$, where $N(r, D)$ denotes the counting function for D as given in [**1** or **7**]. We easily obtain the *First Main Theorem*,

$$N(r, \varphi, f) + m(r, \varphi, f) = T(r, f) + T(r, \varphi) + c,$$

where the *proximity term* $m(r, \varphi, f) \geq 0$. Our main result is the following

SECOND MAIN THEOREM. *Let f, $\varphi_1, \ldots, \varphi_q$ be distinct meromorphic functions on $\mathbf{C}^n$ $(q \geq n-1)$ such that*

(i) $\operatorname{rank}(\varphi_1, \ldots, \varphi_q) = n - 1$,

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(ii) $\operatorname{rank}(f, \varphi_1, \ldots, \varphi_{n-1}) = n$.
Then

$$\| \quad (q-2)T(r,f) \leq \sum_{j=1}^{q} N(r,\varphi_j,f) - N(r, R(f,\varphi_1,\ldots,\varphi_{n-1})) + O\left(\sum_{j=1}^{q} T(r,\varphi_j) + \log rT(r,f)\right).$$

Here "rank" means the maximal rank of the derivative matrix, and R stands for the ramification divisor (given by the Jacobian determinant). The symbol $\|$ means that the inequality is valid for all $r > 0$ outside a set of finite Lebesgue measure. (If f is of finite order and the φ_j are rational, then the inequality of the theorem is valid for all $r > 0$.)

As in the classical theory, we let $\overline{N}(r,\varphi,f)$ denote the counting function obtained by reducing all multiplicities to 1. The Second Main Theorem can be restated as follows:

COROLLARY 1. *Let f, $\varphi_1, \ldots, \varphi_q$ be distinct meromorphic functions such that*

$$\operatorname{rank}(f,\varphi_1,\ldots,\varphi_q) = \operatorname{rank}(\varphi_1,\ldots,\varphi_q) + 1.$$

Then

$$\| \quad (q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}(r,\varphi_j,f) + O\left(\sum_{j=1}^{q} T(r,\varphi_j) + \log rT(r,f)\right).$$

For example, if $\varphi_j = \varphi_j(z_1,\ldots,z_{n-1})$ and $\partial f/\partial z_n \not\equiv 0$, then f, $\varphi_1,\ldots,\varphi_q$ satisfy the hypothesis of Corollary 1. More generally, we can let $\varphi_j = \varphi_j(g_1,\ldots,g_p)$ where $p \leq n-1$ and the g_k are meromorphic functions on $\mathbf{C}^n$ such that $\operatorname{rank}(f,g_1,\ldots,g_p) = p+1$.

If $T(r,\varphi) = o(T(r,f))$, then we define the defect $\delta(\varphi,f)$ as in the one variable case above, and we similarly let $\Theta(\varphi,f) = \liminf[1 - \overline{N}(r,\varphi,f)/T(r,f)]$. As in the classical case, we have $0 \leq \delta(\varphi,f) \leq \Theta(\varphi,f) \leq 1$. We now state our general defect relation, which follows immediately from Corollary 1.

COROLLARY 2 (DEFECT RELATION). *Let f, $\varphi_1,\ldots,\varphi_q$ be as in Corollary 1. If $T(r,\varphi_j) = o(T(r,f))$ for $1 \leq j \leq q$, then*

$$\sum \delta(\varphi_j,f) \leq \sum \Theta(\varphi_j,f) \leq 2.$$

The proof of our Second Main Theorem uses the methods of [1 and 5] and the essential estimate given in the lemma below. We let

$$\omega = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(|w^0|^2 + |w^1|^2)$$

denote the Fubini-Study 2-form on $\mathbf{CP}^1$, and we let

$$\rho(a,b) = \frac{|a^1b^0 - a^0b^1|}{(|a^0|^2 + |a^1|^2)^{1/2}(|b^0|^2 + |b^1|^2)^{1/2}}$$

denote the chordal distance on $\mathbf{CP}^1$. We consider the function

$$\gamma = \rho^{-2}(4q - 2\log\rho)^{-2} \text{ on } \mathbf{CP}^1 \times \mathbf{CP}^1$$

(γ blows up along the diagonal). Let f, $\varphi_1, \dots, \varphi_q$ be as in the Second Main Theorem, and assume $q \geq n+2$. We regard f, φ_j as meromorphic maps into $\mathbf{CP}^1$. Let

$$S = \operatorname{supp}\left[\sum_{j=1}^{q} D(\varphi_j, f) + R(f, \varphi_1, \dots, \varphi_{n-1})\right].$$

We define the volume form Ψ on $\mathbf{C}^n - S$ (which is a variant of the volume form given by Carlson and Griffiths [1]) by

$$\Psi = \left[\prod_{j=1}^{q} \gamma(\varphi_j, f)\right] f^*\omega \wedge \varphi_1^*\omega \wedge \cdots \wedge \varphi_{n-1}^*\omega.$$

Recall that the Ricci form $\operatorname{Ric}\Psi$ is given by $\operatorname{Ric}\Psi = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log h$ where $\Psi = h(\sqrt{-1}\,dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (\sqrt{-1}\,dz_n \wedge d\bar{z}_n)$. We let

$$\theta = \operatorname{Ric}\Psi + 2\sum_{i=1}^{n-1} \varphi_1^*\omega \quad \text{on } \mathbf{C}^n - S.$$

LEMMA. *θ is positive and $\theta^n > \lambda^{2q-2}\Psi$ on $\mathbf{C}^n - S$, where*

$$\lambda = \min_{j \neq k} \rho(\varphi_j, \varphi_k).$$

The positivity of θ is easy to verify without any condition on the rank of the φ_j. However the volume-form inequality of the lemma is not true in general if $\operatorname{rank}(\varphi_1, \dots, \varphi_q) = n$. Details will appear in [6].

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