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A NONLINEAR PARTIAL DIFFERENTIAL EQUATION AND THE UNCONDITIONAL CONSTANT OF THE HAAR SYSTEM IN L^p

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1. Introduction. Our aim here is to identify the best constant in an inequality (see Theorem 1) that has proved useful in the study of singular integrals, stochastic integrals, the structure of Banach spaces, and in several other areas of study. Our work yields the unconditional constant of the Haar system in $L^p(0,1)$ and rests partly on solving the nonlinear partial differential equation

$$(p-1)[yF_y - xF_x]F_{yy} - [(p-1)F_y - xF_{xy}]^2 + x^2F_{xx}F_{yy} = 0 \tag{1}$$

for F nonconstant and satisfying other conditions on a suitable domain of $\mathbf{R}^2$.

We assume throughout that $1 < p < \infty$ and write L^p for the real Lebesgue space $L^p(0,1)$. The *unconditional constant* $K_p(e)$ of a sequence $e = (e_1, e_2, \ldots)$ in L^p is the least $K \in [1, +\infty]$ with the property that if n is a positive integer and $a_1, \ldots, a_n$ are real numbers such that $\|\sum_{k=1}^n a_k e_k\|_p = 1$, then

$$\left\| \sum_{k=1}^{n} \epsilon_k a_k e_k \right\|_p \le K$$

for all choices of signs $\epsilon_k \in \{-1, 1\}$. The sequence e is a *basis* of L^p if, for every $f \in L^p$, there is a unique sequence a such that $\|f - \sum_{k=1}^n a_k e_k\|_p \to 0$ as $n \to \infty$. A sequence $d = (d_1, d_2, \ldots)$ in L^p is a *martingale difference sequence* if d_{n+1} is orthogonal to $\varphi(d_1, \ldots, d_n)$ for all bounded continuous functions $\varphi \colon \mathbf{R}^n \to \mathbf{R}$ and all $n \ge 1$.

The Haar system $h = (h_1, h_2, \ldots)$ is both a basis of L^p (Schauder [**11**]) and a martingale difference sequence: h_{n+1} is supported by a set on which $\varphi(h_1, \ldots, h_n)$ is constant. By an inequality of Paley (see [**10** and **6**]), its unconditional constant $K_p(h)$ is finite. More generally [**2**], $\sup_d K_p(d)$ is finite where the supremum is taken over all martingale difference sequences d in L^p. In fact, if d is a martingale difference sequence and e is a basis of L^p, then

$$K_p(d) \le K_p(h) \le K_p(e). \tag{2}$$

The right-hand side of (2) is due to Olevskiĭ[**8, 9**] and the left-hand side to Maurey [**7**]. Lindenstrauss and Pełczyński [**5**] have an alternative approach to

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the right-hand side via Liapounoff's theorem on the range of a vector measure. An alternative proof of the left-hand side follows easily from a result of Aldous [**1**]; also see [**4**].

Let $p^* = p \vee p'$, the maximum of p and p', where $1/p + 1/p' = 1$.

THEOREM 1. *The inequality*

$$\left\| \sum_{k=1}^{n} \epsilon_k d_k \right\|_p \leq (p^* - 1) \left\| \sum_{k=1}^{n} d_k \right\|_p \tag{3}$$

holds for all martingale difference sequences d in L^p, $\epsilon_k \in \{-1, 1\}$, and $n \geq 1$. The constant $p^* - 1$ is best possible. Furthermore, strict inequality holds in (3) *if and only if $p \neq 2$ and $\| \sum_{k=1}^{n} d_k \|_p > 0$.*

Apart from the best constant, inequality (3) was first proved for the special case $d_k = a_k h_k$ by Paley [**10**] and, for all martingale difference sequences d, by the author in [**2**], where it was also shown that the numbers ϵ_k may be replaced by measurable functions v_k with values in $[-1, 1]$ for those d for which d_{k+1} is orthogonal to every bounded continuous function of $v_1, \ldots, v_{k+1}, d_1, \ldots, d_k$. The inequality then holds with the same constant [**3**].

COROLLARY 1. *The unconditional constant of the Haar system in L^p is given by*

$$K_p(h) = p^* - 1. \tag{4}$$

2. Sketch of the proof of Theorem 1. Consider the domain

$$\Omega = \left\{ (x, y, t) \in \mathbf{R}^3 : \left| \frac{x - y}{2} \right|^p < t \right\},$$

with boundary $\partial\Omega$, and note, for example, that the section $\{(x, t) : (x, y, t) \in \Omega\}$ determined by y is convex. The following lemma is an immediate consequence of Theorem 3.3 of [**3**].

LEMMA 1. *If $u : \Omega \cup \partial\Omega \to \mathbf{R}$ is continuous,*

(i) *for all $y \in \mathbf{R}$, the mapping $(x, t) \to u(x, y, t)$ is convex on the section of Ω determined by y,*

(ii) *for all $x \in \mathbf{R}$, the mapping $(y, t) \to u(x, y, t)$ is convex on the section of Ω determined by x, and*

(iii) *for all $(x, y, t) \in \partial\Omega$,*

$$u(x, y, t) \leq \left| \frac{x + y}{2} \right|^p,$$

then

$$u(0, 0, 1) \left\| \sum_{k=1}^{n} \epsilon_k d_k \right\|_p^p \leq \left\| \sum_{k=1}^{n} d_k \right\|_p^p.$$

Here we seek the greatest function u satisfying the conditions of this lemma. Such a function does exist and must also satisfy the symmetry property

$$u(x,y,t) = u(y,x,t) = u(-x,-y,t)$$

and the homogeneity property

$$u(x,y,t) = \lambda^{-p} u(\lambda x, \lambda y, \lambda^p t), \qquad \lambda > 0.$$

Therefore, if $F(x,y) = u(x,y,1)$ and $(x,y,t) \in \Omega$, then

$$u(x,y,t) = tF(xt^{-1/p}, yt^{-1/p}). \tag{5}$$

Now suppose that u is twice continuously differentiable on a neighborhood of some point $(x_0, y_0, 1) \in \Omega$. Then (i) and (ii) imply that, on the same neighborhood, $u_{xx} \geq 0$, $u_{yy} \geq 0$, $u_{tt} \geq 0$, $u_{xx}u_{tt} - u_{xt}^2 \geq 0$, and $u_{yy}u_{tt} - u_{yt}^2 \geq 0$. These lead, by (5), to the following system of differential inequalities for F on a neighborhood of (x_0, y_0):

$$F_{xx} \geq 0, \tag{6}$$

$$F_{yy} \geq 0, \tag{7}$$

$$x^2 F_{xx} + 2xyF_{xy} + y^2 F_{yy} - (p-1)[xF_x + yF_y] \geq 0, \tag{8}$$

$$(p-1)[xF_x - yF_y]F_{xx} - [(p-1)F_x - yF_{xy}]^2 + y^2 F_{xx}F_{yy} \geq 0, \tag{9}$$

$$(p-1)[yF_y - xF_x]F_{yy} - [(p-1)F_y - xF_{xy}]^2 + x^2 F_{xx}F_{yy} \geq 0. \tag{10}$$

The maximality of u also implies that

$$F(x,y) = \left|\frac{x+y}{2}\right|^p, \qquad (x,y) \in \partial D, \tag{11}$$

where $D = \{(x,y)\colon |x-y| < 2\}$, and suggests that, on some subdomains of D, equality should hold in at least one of the above differential inequalities, which one depending upon the subdomain.

Additional study leads to the consideration of the differential equation (1) on the subdomain

$$\{(x,y) \in D\colon x > 0, (1-2/p)x < y < x\}. \tag{12}$$

The key step is to solve equation (1) on (12) for a function F in harmony with the boundary condition (11) and the other requirements of our problem.

Equation (1) has the following solution on (12):

$$F = (w-1)^p \tag{13}$$

where $w > p$ is the unique solution to

$$x^p[1 - p(x-y)/2x] + pw^{p-1} - w^p = 0. \tag{14}$$

If $1 < p \leq 2$, this function F not only satisfies (10) with equality on the subdomain (12) but also satisfies (6)–(9) there.

On the subdomain

$$\{(x,y) \in D\colon x > 0, -x < y < (1-2/p)x\}, \tag{15}$$

the differential equations corresponding to (8)–(10) have the solution

$$F(x,y) = \left|\frac{x+y}{2}\right|^p + \left[1 - \left|\frac{x-y}{2}\right|^p\right](p-1)^p. \tag{16}$$

If $1 < p \leq 2$, then (6) and (7) are also satisfied on (15).

LEMMA 2. *Let $1 < p \leq 2$ and u be the continuous function on $\Omega \cup \partial\Omega$ satisfying (5) on Ω where F is the continuous function on D given by (13) on the subdomain (12), by (16) on (15), and satisfying $F(x,y) = F(y,x) = F(-x,-y)$ on D. Then u satisfies the conditions of Lemma 1 and is, in fact, the greatest such function.*

In particular, $u(0,0,1) = (p-1)^p$ so that, by Lemma 1,

$$(p-1)\left\|\sum_{k=1}^{n} \epsilon_k d_k\right\|_p \leq \left\|\sum_{k=1}^{n} d_k\right\|_p.$$

Using the identity $(p-1)(p'-1) = 1$, we obtain

$$\left\|\sum_{k=1}^{n} \epsilon_k d_k\right\|_p \leq (p'-1)\left\|\sum_{k=1}^{n} d_k\right\|_p,$$

which is the inequality of Theorem 1 in the case $1 < p \leq 2$. The case $2 < p < \infty$ follows by duality [2].

To see that the constant $p^* - 1$ is best possible, consider the following example. Let $1 < p \leq 2$ and $x > 0$. Let $w > p$ satisfy $x^p + pw^{p-1} - w^p = 0$. Set $\theta = 1 - 1/w = 1/w'$ and

$$\beta_k = 1 - \frac{w\delta}{x + k\delta}, \qquad k \geq 1,$$

where $0 < \delta < x/w$. Using the same notation for an interval $[a,b)$ and its characteristic function, set

$$\begin{aligned}
d_1 &= x[0,1),\\
d_2 &= \delta[0,\beta_1) + [\theta(x+\delta) - x][\beta_1, 1),\\
d_3 &= \delta[0,\beta_1\beta_2) + [\theta(x+2\delta) - (x+\delta)][\beta_1\beta_2, \beta_1),
\end{aligned}$$

and so forth. Then

$$\lim_{x\to 0}\lim_{\delta\to 0}\lim_{n\to\infty}\left\|\sum_{k=1}^{n} (-1)^k d_k\right\|_p = 1$$

and

$$\lim_{x\to 0}\lim_{\delta\to 0}\lim_{n\to\infty}\left\|\sum_{k=1}^{n} d_k\right\|_p = p-1.$$

For the complete proof of Theorem 1 and the study of related inequalities and boundary value problems, see [4].

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