mixture of standard theory and new research work which has not previously appeared in book form. It is a good textbook for mathematicians and physicists who want to learn the C^*-quantum physics. In the following, I will review the book chapter by chapter. It consists of two volumes. The first volume is devoted to mathematical theory of operator algebras and their dynamics, and the second to its applications to quantum statistical mechanics, and to the models of quantum statistical mechanics. The first volume contains four chapters and the second contains two chapters. The chapters of the second volume are numbered consecutively with those of the first. Chapter 1 is a brief historical introduction. The historical introduction is not concerned with the theory of operator algebras, but it is concerned with the interplay between operator algebras and quantum physics. Chapter 2 is C^*-algebras and von Neumann algebras. It discusses the elementary theory of operator algebras, Tomita-Takesaki theory and the standard form of von Neumann algebras, quasilocal algebras, and miscellaneous results and structure. The authors select material from the general theory of operator algebras which is needed for quantum physics. Chapter 3 is groups, semigroups, generators. In this chapter, the authors discuss mainly derivations, automorphism groups and generation problems. Chapter 4 is decomposition theory. Here various decompositions of states are treated. The authors use a modern method developed recently by many researchers, which combines the reduction theory of von Neumann with the Choquet theory. In the ergodic decomposition which is of importance in mathematical physics, the notion of G-abelianness introduced by Lanford and Ruelle is used.

The contents of the first volume is rich enough to use it as a textbook for advanced graduate students in the field of functional analysis. For physics students, there might be too much abstraction. Chapter 5 is states in quantum statistical mechanics. Here the authors describe continuous quantum systems, KMS states, and stability and equilibrium. The material prepared in Volume 1 is seriously used in the two sections of KMS states, and stability and equilibrium. Chapter 6 is models of quantum statistical mechanics. Here, the authors describe quantum spin systems and continuous quantum systems. This chapter is most instructive for C^*-algebraists, though there is little involvement of the theory of operator algebras. The reason is that it would be an interesting problem to extend various results in this chapter to more general C^*-dynamics.

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Global Lorentzian geometry, by John K. Beem and Paul E. Ehrlich, Pure and Applied Mathematics, vol. 67, Dekker, New York, 1981, vi + 460 pp.,

The past two decades have witnessed an enormous growth in the development of global methods in Lorentzian geometry. The time seems ripe for a systematic treatment of global Lorentzian geometry written in the language of modern differential geometry by mathematicians for mathematicians acquainted with Riemannian geometry and interested in the geometry of general relativity. Responding to this need, the authors have written a lucid and mathematically precise account of a significant portion of the global theory of Lorentzian manifolds, frequently offering a fresh and illuminating perspective to what now has become standard material. In addition, their monograph includes a well integrated and expanded presentation of many of their own results which have appeared in the literature in recent years, as well as some new results which have not. Those uninitiated in Lorentzian geometry will especially appreciate the authors' painstaking effort to point out the similarities and differences between Lorentzian and Riemannian geometry. Here we touch upon some of these similarities and differences and, also, briefly discuss the issue of singularities in general relativity which, to a large extent, provided the stimulus for the surge of activity in global Lorentzian geometry.

A Lorentzian manifold is a smooth manifold M equipped with a Lorentzian metric g which is a smooth assignment of a nondegenerate bilinear form $g|_p$: $T_pM \times T_pM \to \mathbf{R}$ with diagonal form $(-, +, \cdots, +)$ to each tangent space T_pM of M. Thus each tangent space of a Lorentzian manifold (M, g) is naturally isometric to Minkowski space, the space-time of special relativity. A distinguishing feature of a Lorentzian manifold is that it admits a *causal structure*. The Lorentzian metric g divides the vectors at each point into three classes: a vector X is said to be timelike, spacelike, or null according to whether $g(X, X)$ is negative, positive, or zero, respectively. (X is said to be nonspacelike if it is timelike or null.) The null vectors in T_pM form a double cone which is filled by the timelike vectors. A curve in M is said to be timelike, spacelike, or null if its tangent at each point is timelike, spacelike, or null, respectively. In order to discuss the causal structure of (M, g) in the large it is necessary that (M, g) be *time orientable*, i.e. that there exist a continuous assignment of a *future* direction at each point of M. If M is not time orientable, it at least admits a two sheeted covering which is. In a *space-time* (i.e. a time oriented Lorentzian manifold) the following two causal relations can be defined. For points p, q in M, the point q is said to be in the chronological future of p, written $p \ll q$, if there is a future directed timelike curve from p to q, and q is said to be in the causal future of p, written $p \leqslant q$, if $q = p$ or if there is a future directed nonspacelike curve from p to q. The global causal structure of (M, g) is determined by the properties of the causal relations "$\ll$" and "$\leqslant$". Of special importance is the fact (which has been exploited by Penrose [9]) that the causal structure of a space-time is conformally invariant.

In general relativity the actual space-time universe in which we live is modeled by a four dimensional Lorentzian manifold. In this way the local accuracy of special relativity is built into the theory. The timelike curves describe the histories (or world lines) of material particles or "observers", with the timelike *geodesics* representing the worldlines of *freely falling* observers. The null geodesics represent the histories of light rays. The proper time (i.e. arc length) along a timelike curve which describes the world line of some observer corresponds to the time kept by a suitably physical (e.g. atomic) clock carried by the observer. The metric g is assumed to satisfy the Einstein field equation which relates the geometry of space-time to the energy-momentum content of the universe. The Einstein equation may be thought of as a generalization of Poisson's equation for the gravitational potential in Newtonian theory.

To the Riemannian geometer perhaps no assumption is more natural than that of geodesic completeness. The situation in Lorentzian geometry, however, is quite different. Many of the space-times which are exact solutions to the Einstein equations are timelike geodesically incomplete. For instance the Friedmann models which are the classical, and to this day standard, cosmological models based on general relativity all predict a fantastic "big bang" beginning to the universe. All the timelike geodesics are past incomplete, with the mass density (and hence curvature) becoming infinite in a finite proper time along each such geodesic. In the (extended) Schwarzschild solution, which today is understood to represent the geometry of spherically symmetric gravitational collapse, the world line of any observer (freely falling or not) who has entered into the "black hole" region by crossing the "event horizon" at the Schwarzschild radius $r = 2m$ (where m is the mass of the black hole) is future incomplete. These classical solutions, although constructed to model important physical situations, are based on stringent symmetry assumptions (e.g. the Friedmann models are spatially isotropic). It was long felt by many (including Einstein) that the "singular nature" of these exact solutions (as is evidenced by their incompleteness) might be due to their extreme symmetry. Perhaps less idealized models which allow for inhomogeneities don't exhibit such singular behavior.

The question as to whether or not such singular behavior is inherent in the theory of general relativity was resolved by a remarkable series of papers published in the second half of the 1960's beginning with the innovative paper of Penrose [**10**] and culminating with the triumphant paper of Hawking and Penrose [**6**]. The results of these papers show that the incompleteness exhibited in the idealized exact solutions is *generic*, i.e. there are generic classes of physically relevant space-times which are either timelike *or* null geodesically incomplete. Unfortunately, even today not a great deal is known about the nature or structure of such incompleteness. As is well known, a *Riemannian* manifold is geodesically complete if and only if it is complete as a metric space (with distance function d, defined by $d(p, q) = \inf\{\text{length of } \gamma\text{: } \gamma \text{ a curve from } p \text{ to } q\}$). Thus if an inextendible Riemannian manifold M is geodesically incomplete then the points added to M to obtain its Cauchy completion may be viewed as its singular points. Unfortunately, there is no such tidy mathematical interpretation of geodesic incompleteness in the Lorentzian case since the indefinite metric of a Lorentzian manifold M cannot be used in the same way to endow M with a metric space structure. In contrast to the Riemannian case, there are examples of *compact* Lorentzian manifolds which are geodesically incomplete (see [**7**]). Procedures have been devised for attaching a singularity boundary to an incomplete space-time but no such procedure has yet proven to be completely satisfactory. At any rate, the physical interpretation of an incomplete timelike geodesic is clear: there is a freely falling observer with either a finite past or a finite future (as measured by the observer). The physical significance of an incomplete null geodesic is not so apparent since its affine parameters do not correspond to local time as measured by some observer.

The global analysis of Lorentzian manifolds, much of which was developed during the period in which these generic singularity theorems appeared, focuses attention on the causal structure of a space-time and the behavior of its timelike and null geodesics. Although there are significant differences, much of the treatment of the structure of timelike geodesics is analogous to the treatment of geodesics in a Riemannian manifold. For example one can show by methods quite similar to the Riemannian case that timelike geodesics locally *maximize* arc length. The reason that they locally maximize (instead of minimize as in the Riemannian case) boils down to the fact that future directed timelike vectors obey the "reverse triangle inequality": for future directed timelike vectors A, B, $\|A + B\| \geqslant \|A\| + \|B\|$ (where $\|A\| = [-\langle A, A\rangle]^{1/2}$). (It is worth noting that in Lorentzian manifolds having dimension $\geqslant 3$, spacelike geodesics are neither locally maximizing nor locally minimizing.)

An important element in the proofs of many of the singularity theorems is knowing when a timelike (or null) geodesic maximizes *in the large* in some sense. Let p and q be points in a space-time M with q in the causal future of p. Motivated by the Riemannian case, a nonspacelike geodesic from p to q is said to be maximal if its length is greater than or equal to the length of any other nonspacelike curve from p to q. It can be shown, just as for minimal curves in a Riemannian manifold, that a maximal curve from p to q is a geodesic. The well-known theorem of Hopf and Rinow ensures that if a *Riemannian* manifold is geodesically complete then any two distinct points can be joined by a minimal geodesic. The Lorentzian analogue of this result is false. Indeed, there are examples of geodesically complete space-times for which there are points p, q with $p \ll q$ such that there are no timelike geodesics whatsoever from p to q (e.g. anti-de Sitter space; see [**2**] and [**11**]). Seeking conditions which ensure the existence of maximal geodesics in a space-time M, one expects causality conditions to come into play. For instance, there is obviously no maximal geodesic joining two points of a *closed* timelike curve in M. (The existence of such a curve signifies within the context of general relativity the most flagrant type of causality violation since it implies the ability of some observer to communicate with his past.) The *strong causality condition* asserts that there can be no closed or even "almost closed" timelike curves in M. As was proved by Seifert ([**12**, 1967]; see also Avez [**1**, 1963]) the (now standard) condition in Lorentzian geometry which ensures that causally related points can be joined by a maximal geodesic is *global hyperbolicity*. A space-time M is said to be globally hyperbolic if it is strongly causal and if the sets of the form $J^+(p) \cap J^-(q)$ are compact in M (where $J^+(p) \equiv$ causal future of $p = \{x \in M: p \leqslant x\}$ and $J^-(q) \equiv$ causal past of q). A classical theorem of causal theory due to Geroch [**3**] says that a space-time M is globally hyperbolic if and only if it admits a *Cauchy surface*. A Cauchy surface is a set which is intersected by each inextendible nonspacelike curve in M once and only once. It is necessarily a codimension 1 topological submanifold of M. For an in depth account of the Cauchy surface concept and the closely related concepts of the domain of dependence and the Cauchy horizon—topics which are not treated in great detail in Beem and Ehrlich—see, for example, Hawking and Ellis [**5**].

The calculus of variations of timelike and null geodesics also plays an important role in singularity theory. One can derive formulas for the first and second variations of arc length along timelike curves and introduce the notion of conjugate points along timelike geodesics via Jacobi fields in a manner very similar to what is done in Riemannian geometry. (To our knowledge the first discussion and application of the second variation of arc length along timelike curves to global problems in general relativity was given by Avez [**1**].) One can then show, for instance, by methods analogous to the Riemannian case, that a timelike geodesic cannot maximize beyond its first conjugate point. Beem and Ehrlich give a definitive treatment of the Morse index theory of timelike and null geodesics, and include a discussion of the timelike path space of a globally hyperbolic space-time. The index theory of null geodesics is considerably more delicate. Part of the difficulty stems from the fact that a null vector has square norm zero and, hence, lies in its own orthogonal space.

To illustrate how these global techniques come into play in singularity theory we consider an extremely simplified version of the Hawking-Penrose theorem [**6**].

PROPOSITION. *Suppose M admits a* compact *Cauchy surface V. Suppose, in addition that*

(1) $\mathrm{Ric}(X, X) \geq 0$ *for all nonspacelike vectors X on M, and*

(2) $\mathrm{Ric}(d\gamma/ds, d\gamma/ds) > 0$ *at some point of each inextendible nonspacelike geodesic. Then M is either timelike or null geodesically incomplete.* (*Here,* $\mathrm{Ric}(X, X) = \sum_{i,j} R_{ij} X^i X^j$, *where R_{ij} are the components of the Ricci tensor.*)

When interpreted within the context of general relativity the curvature conditions are referred to as energy conditions. Roughly, condition (1) expresses the nonnegativity of mass-energy density and condition (2) asserts that each nonspacelike geodesic encounters some matter. Briefly, the proof proceeds as follows. Let η be an arbitrary future directed inextendible timelike curve in M which intersects V. Let q_i be a sequence of points extending indefinitely into the future of V along η and let p_i be a sequence of points extending indefinitely into the past of V along η. Since M is globally hyperbolic there is a maximal timelike geodesic segment γ_i from p_i to q_i for each i. Furthermore, since V is Cauchy, it can be easily shown that each segment γ_i must intersect V. Using the compactness of V one can obtain an inextendible limit curve γ of the γ_i, each finite segment of which is maximal. Hence γ is a timelike, or possibly null, geodesic *line*, and in particular cannot contain any pairs of conjugate points. On the other hand if we assume that γ is a *complete* geodesic then standard index form techniques can be used (in the spirit of Myers [**8**]) together with the curvature assumptions (which have a focusing effect on nonspacelike geodesics) to show that γ must contain a pair of conjugate points. The conclusion must be that γ is incomplete. The proposition is analogous to the result of Gromoll and Meyer [**4**] that a complete Riemannian manifold with positive Ricci curvature has only one end.

It should be mentioned that, historically, mathematical relativists developed an alternative method for predicting the occurrence of conjugate points (or focal points). This method makes use of the so-called Raychaudhuri equation, which is a formula for the rate of change of the expansion of a congruence of timelike (or null) geodesics and which can sometimes be used to predict the occurrence of conjugate points and focal points in situations where index methods fail. Beem and Ehrlich give a careful treatment of Raychaudhuri's equation and its applications via the concept of *Jacobi tensors* which they discuss in a neat coordinate and frame independent manner.

The proof of the general Hawking-Penrose theorem still entails establishing the existence of a nonspacelike geodesic line in a space-time which, by the hypotheses (and the negation of the conclusion), is strongly causal, but which needn't be globally hyperbolic. Beem and Ehrlich give an especially beautiful treatment of this result. The Hawking-Penrose theorem essentially falls out of a general exposition on the existence of nonspacelike geodesic lines and rays which occupies one chapter of their book. Since they obtain results about space-times which are assumed to be strongly causal but not necessarily globally hyperbolic, the existence problem is substantially more complicated than the corresponding problem in Riemannian geometry. The proofs in this chapter make significant use of the *Lorentzian* distance function d which is defined as follows,

$$d(p,q) = \begin{cases} 0 & q \notin J^+(p), \\ \sup\{\text{length of } \gamma\} \end{cases}$$

where the sup is taken over all future directed nonspacelike curves γ from p to q. Of course this is not a distance function in the sense of metric spaces, but it does have a few nice properties which the authors judiciously exploit. In fact the authors present a detailed study of the Lorentzian distance function and, in particular, derive some interesting results relating properties of the Lorentzian distance function to the causal structure of space-time.

Many other topics, in addition to those considered above, are covered in the book under review, such as the causality and completeness of warped product spaces, the structure of two-dimensional space-times, the stability of Robertson-Walker spaces, the Lorentzian cut locus, and Lorentzian comparison theorems. Included among the appendices is a geometric explication of the so-called *generic* curvature condition and a presentation of Harris' Lorentzian version of Toponogov's triangle comparison theorem. All topics are treated in an invariant manner and are written with impressive clarity and precision.

The global theory of Lorentzian geometry has "grown up" during the last twenty years, and Beem and Ehrlich have given us an authoratative and highly readable treatment of the subject as it stands today.

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The Hardy-Littlewood method, by R. C. Vaughan, Cambridge Tracts in Mathematics, vol. 80, Cambridge University Press, Cambridge, 1981, xii + 172 pp., $34.50.

A few years ago I heard a prominent algebraic number theorist exclaim: "What, the Hardy-Littlewood method is still alive? I thought it had been dead long ago". The book under review shows that the method is alive and well!

Let $\mathfrak{F}\colon \mathfrak{D} \to \mathbf{Z}$ be a map into the integers assuming each value at most finitely often. The number $N = N(\mathfrak{F}, \mathfrak{D})$ of zeros of $\mathfrak{F}$ is the constant term of the formal series

$$F(z) = \sum_{\mathbf{x} \in \mathfrak{D}} z^{\mathfrak{F}(\mathbf{x})}.$$

Assuming that F is analytic in the disk $|z| < 1$ with the possible exception of $z = 0$, we may invoke Cauchy's integral formula to obtain

$$N = \frac{1}{2\pi i} \int_C z^{-1} F(z)\, dz, \tag{1}$$

where C is a circle centered at 0 with radius $\rho < 1$. What is surprising is not this formula, but the way in which the integral on the right may often be evaluated or approximated so as to give information about diophantine problems.

Hardy and Ramanujan [**1918**] used this integral formula to obtain an asymptotic relation for the partition function, and to deal with the number of representations of integers by sums of squares. More generally, in a series of papers beginning in 1920, Hardy and Littlewood applied the formula to Waring's problem, i.e. the representation of integers n by sums of nonnegative kth powers:

$$n = x_1^k + \cdots + x_s^k. \tag{2}$$

Here $\mathfrak{D} = \mathbf{Z}^+ \times \cdots \times \mathbf{Z}^+$ where $\mathbf{Z}^+$ are the nonnegative integers, $\mathfrak{F}(\mathbf{x}) = \mathfrak{F}(x_1, \ldots, x_s) = x_1^k + \cdots + x_s^k - n$, and $N = N(k, s, n)$. Hardy and Ramanujan noted that in the case $k = 2$, i.e. in the case of squares, the integrand in (1)