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# EXOTIC CLASSES FOR MEASURED FOLIATIONS

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A measured foliation $(F, \mu)$ is a $C^2$-foliation $F$ on a smooth manifold $M$ and a transverse invariant measure $\mu$ for $F$ [14]. Inspired by the foliation index theorem of Connes [4, 5], we study the result of integrating *normal data* to $F$ over the leaf space $M/F$. This produces new secondary-type exotic classes for measured foliations [7]. These classes have applications to $SL_q$-foliations, to the study of groups of volume-preserving diffeomorphisms, and also are useful for relating the geometry of $F$ to the values of the usual secondary classes [8, 9].

THEOREM 1. *Let $(F, \mu)$ be a measured foliation of codimension $q$ on $M$. If either $M$ is closed and orientable, or $\mu$ is absolutely continuous (so it is represented by a closed form $d\mu$), then there is a well-defined characteristic map*

$$\chi_\mu: H^*(\mathrm{gl}_q, O_q) \longrightarrow H^{*+q}(M).$$

We call the image of $\chi_\mu$ the $\mu$-classes of $(F, \mu)$.

For $M^m$ compact and $y_I \in H^n(\mathrm{gl}_q, O_q)$, the class $\chi_\mu(y_I)$ is defined as the geometric current in $H_{m-n-q}(M)$ obtained by integrating over the leaf space of $F$, via $\mu$, the leaf classes corresponding to $y_I$. Duality then produces the invariant in $H^{n+q}(M)$. If $d\mu$ is a closed form representing $\mu$, then a cocycle representing $\chi_\mu(y_I)$ is $\Delta(y_I) \cdot d\mu$, where $\Delta: WO_q \longrightarrow A^{\cdot}(M)$ is the secondary map for $F$, [2, 10]. Complete details and properties of $\chi_\mu$ are described in [7].

The values of the $\mu$-classes depend on the measure $\mu$ and the dynamical behavior of $F$ in a neighborhood of the support of $\mu$. It is conjectured that subexponential growth of the leaves of $F$ implies the $\mu$-classes vanish; this can be shown in some cases. Examples can be constructed for which all of the $\mu$-classes are nontrivial.

The canonical measure associated to an $SL_q$-foliation $(F, \omega)$—where $\omega$ is a transverse invariant volume form—defines a characteristic map $\chi_\omega: H^*(\mathrm{sl}_q, SO_q) \longrightarrow H^{*+q}(M)$, and these come from universal classes for the Haefliger classifying space $B\Gamma_{SL_q}$. There are additional $\mu$-classes for measured foliations with framed normal bundles, and corresponding universal classes for $B\overline{\Gamma}_{SL_q}$, the homotopy fiber of $B\Gamma_{SL_q} \longrightarrow BSL_q$.

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THEOREM 2. *The characteristic maps*

$$\chi\colon H^n(\mathrm{sl}_q, SO_q) \longrightarrow H^{n+q}(B\Gamma_{SL_q}),$$

$$\chi^s\colon H^n(\mathrm{sl}_q) \longrightarrow H^{n+q}(B\overline{\Gamma}_{SL_q})$$

*are injective for $n < [(q-1)/4]$. Further, $\chi^s$ is injective on the image $H^*(\mathrm{so}_q) \longrightarrow H^*(\mathrm{sl}_q)$.*

To show nontriviality, we compute the values of the $\mu$-classes for the foliations obtained by suspending the action of $SL_qZ$ on $T^q$. This type of example was suggested to us by W. Thurston. The corresponding characteristic map is related to the Van Est map of continuous cohomology $H^*(\mathrm{sl}_q, SO_q) \longrightarrow H^*(SL_qZ; \mathbf{R})$, which is injective in degrees less than $[(q-1)/4]$ by Borel [1]. Note that $\chi^s$ is nontrivial for all $q \geqslant 3$, but Theorem 2 only asserts $\chi$ is nontrivial for $q \geqslant 25$. We do not know of $SL_q$-foliations with small codimension and nontrivial $\mu$-classes from $\chi$.

Let $\omega$ be a volume form on $\mathbf{R}^q$ with infinite total mass and $\mathrm{Diff}^c_\omega\mathbf{R}^q$ the group of compactly supported diffeomorphisms which preserve $\omega$. We use McDuff's generalization of the Mather-Thurston theorem [11] and Theorem 2 to prove

COROLLARY 1.. *There are inclusions of* $\mathbf{Q}$-*vector spaces*

$$H_n(\mathrm{so}_q; \mathbf{R}) \hookrightarrow H_n(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; \mathbf{Q})$$

*for $n < q$, and $\mathbf{R} \subseteq H_3(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; \mathbf{Q})$ for all $q \geqslant 3$.*

It was shown that $H_1(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; Z) = 0$ for $q > 2$ by Thurston-Banyaga. Corollary 1 gives the first nonvanishing results for the group homology in degrees less than $q+1$; in degrees $\geqslant q+1$, the secondary classes of $SL_q$-foliations detect nontrivial homology of $B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q$. McDuff has investigated in [12] the geometrical significance of some of these new invariants for $\mathrm{Diff}^c_\omega\mathbf{R}^q$ and also defined further interesting classes.

The residuable secondary classes are the cocycles $y_Ic_J$ in $H^*(WO_q)$ with degree $c_J = 2q$ maximal. The "integration over the fiber" process is faithful on these classes, so a residue theory can be developed for them. Given a measured foliation $(F, \mu)$ with support $\mu = M$, the residuable classes decompose into the measure class $d\mu$ product with a leaf invariant. This observation can be used to relate the residuable secondary classes with the geometry of $F$.

THEOREM 3. *Let $F$ be a codimension $q$ compact foliation (that is, each leaf of $F$ is compact) on a closed manifold $M$. Each residuable secondary class $\Delta_*(y_Ic_J) \in H^*(M)$ is then zero.*

The idea of the proof is to integrate $\Delta(y_I c_J)$ over $M$, decompose this integral over saturated sets—the *Epstein filtration* of the bad set—where each saturated set has a transverse invariant measure of maximal support. Each integral decomposes into a weighted sum of leaf classes, and then we show the leaf classes for a compact foliation uniformly vanish. Details appear in [8].

For codimension one foliations remarkable progress has been made in relating the geometry of a foliation with its Godbillon-Vey invariant [3, 13]. For higher codimensions, it is expected that a geometric interpretation of the residuable secondary classes can be achieved by utilizing the techniques of the proof of Theorem 3, the residue theorem for foliations [6] and the properties of the $\mu$-classes. Some progress on this problem is given in [9].

ADDED IN PROOF. G. Duminy has recently proved that a codimension-one foliation on a compact manifold with nonvanishing Godbillon-Vey invariant must have a resilient leaf (*L'Invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort,* preprint.)

## REFERENCES

1. A. Borel, *Stable real cohomology of arithmetic groups,* Ann. Sci. École Norm. Sup. 4e 7 (1974), 235–272.

2. R. Bott and A. Haefliger, *On characteristic classes of Γ-foliations,* Bull. Amer. Math. Soc. 78 (1972), 1039–1044.

3. J. Cantwell and L. Conlon, *A vanishing theorem for the Godbillon-Vey invariant of foliated manifolds,* to appear (1981).

4. A. Connes, *Sur la théorie non-commutative de l'integration,* Lecture Notes in Math., vol. 725, Springer-Verlag, Berlin and New York, 1979, pp. 19–143.

5. A. Connes and G. Skandalis, *Théorème de l'indice pour les feuilletages,* C. R. Acad. Sci. Paris Sér. A **292** (1981), 871–876.

6. J. Heitsch, *Flat bundles and residues for foliations,* to appear (1981).

7. S. Hurder, *Global invariants for measured foliations,* to appear (1981).

8. ———, *Vanishing of secondary classes for compact foliations,* to appear (1982).

9. ———, *Growth of leaves and differential invariants of foliations,* in preparation (1982).

10. F. Kamber and P. Tondeur, *Foliated bundles and characteristic classes,* Lecture Notes in Math., vol. 493, Springer-Verlag, Berlin and New York, 1975, pp. 1–294.

11. D. McDuff, *The homology of groups of volume preserving diffeomorphisms,* Ann. École Norm. Sup. (to appear).

12. ———, *Some canonical cohomology classes on groups of volume preserving diffeomorphisms,* Trans. Amer. Math. Soc. (to appear).

13. T. Mizutani, S. Morita and T. Tsuboi, *The Godbillon-Vey classes of codimension one foliations which are almost without holonomy,* Ann. of Math. (2) 113 (1981), 515–527.

14. J. Plante, *Foliations with measure preserving holonomy,* Ann. of Math. (2) 102 (1975), 327–361.

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