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THE RADICAL IN A FINITELY GENERATED P.I. ALGEBRA

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Let R be an associative ring over a commutative ring Λ, $p\{X_1, \ldots, X_e\}$ a polynomial on the free noncommuting variables $X_1, \ldots, X_e$, with coefficients in Λ where one of its coefficient is $+1$. We say that R is a P.I. (polynomial identity) ring satisfying $p\{X_1, \ldots, X_e\}$ if $p(r_1, \ldots, r_e) = 0$ for all $r_1, \ldots, r_e$ in R.

We have the following

THEOREM A. *Let $R = \Lambda\{x_1, \ldots, x_k\}$ be a p.i. ring, where Λ is a noetherian subring of the center $Z(R)$ of R. Then, $N(R)$, the nil radical of R, is nilpotent.*

Recall that $N(R) = \bigcap_P P$ where the intersection runs on all prime ideals of R.

We obtain, as a corollary, by taking Λ to be a field, the following theorem, answering affirmatively the open problem which is posed in **[Pr**, p. 186].

THEOREM B. *Let R be a finitely generated P.I. algebra over a field F. Then, $J(R)$, the Jacobson radical of R, is nilpotent.*

This result, in turn, has the following important consequence.

THEOREM C. *Let $R = F\{x_1, \ldots, x_k\}$ be a finitely generated P.I. algebra over the field F. Then, R is a subquotient of some $n \times n$ matrix ring $M_n(K)$ where K is a commutative F-algebra. Equivalently, there exists an n such that R is a homomorphic image of $G(n, t)$ the ring of t, $n \times n$ generic matrices.*

Kemer, in **[K]**, announced a proof of Theorem B with the additional assumption that $\mathrm{char}(F) = 0$. His proof relies on a result of Razmyslov **[Ra**, Theorem 3] and on certain arguments related to the connection between P.I. ring theory and the theory of representation of the symmetric group S_n over F, char $F = 0$. Both results rely heavily on the assumption that char $F = 0$, so they do not seem to generalize directly to arbitrary F.

The previously best known results concerning Theorem A are in **[Ra**, Theorems 1, 3, **Sc**, Theorem 2].

The proof of Theorem C is a straightforward application of Theorem B and a theorem of J. Lewin **[Le**, Theorem 10].

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We sketch a proof of Theorem B.

A major tool in our proof is the following result of Latyshev [**La**, Proposition 12]: "*Let R be a p.i. F-algebra and $I \subset N(R)$ a finitely generated two sided ideal. Then, I is nilpotent.*" Latyshev's result is originally stated with the additional assumption char $F = 0$, but it is superfluous.

We sketch the proof of Theorem B; the complete proof will appear elsewhere.

We have $N(R) = P_1 \cap \cdots \cap P_t$, where P_i are the minimal prime ideals of R, ordered via

$$\text{p.i.d}(R/P_1) = \cdots = \text{p.i.d}(R/P_m) \underset{\neq}{>} \text{p.i.d}(R/P_{m+1}) \geqslant \cdots \geqslant \text{p.i.d}(R/P_t)$$

where $m \leqslant t$ (if $m = t$, $P_{m+1} \equiv R$). Here p.i.d(S) denotes the minimal size of matrices into which S can be embedded.

Let $\pi(R) = \text{p.i.d}(R/P_1)$, $d(R) = \max\{\text{k.d}(R/P_i) | i = 1, \ldots, m\}$ where k.d(S) is the classical Krull dimension of S. One observes that there exists a $b = b(k, d)$ such that if S is an F-algebra satisfying $p(X_1, \ldots, X_e)$ (of degree d) and $S = F\{y_1, \ldots, y_k\}$ then $\text{k.d}(S) \leqslant b < \infty$. We argue on the ordered pair $\langle \pi(R), d(R) \rangle$ ordered lexicographically, that R is a counterexample to the theorem with minimal $\langle \pi(R), d(R) \rangle$. This will imply that there exists a $\lambda \notin P_1 \cup \cdots \cup P_m$, a finite sum of evaluations of some central polynomial of $\pi \times \pi$ matrices ($\pi \equiv \pi(R)$). Using the result of Latyshev quoted above we may assume that $\lambda \in Z(R)$ and by the minimal choice of R, since $\langle \pi(R/\lambda R), d(R/\lambda R) \rangle < \langle \pi(R), d(R) \rangle$, we get that $\lambda^l R \subseteq N(R)$ for some l. Using Latyshev's result once more we may assume that R_λ, the localization of R with respect to the set $\{\lambda, \lambda^2, \ldots\}$, is Azumaya of rank π^2 over its center. This in turn implies that $\lambda^e R \subseteq Zb_1 + \cdots + Zb_h \equiv A$, where $Z \equiv Z(R)$, $b_i \in R$, $i = 1, \ldots, h$ and A is a ring, for some e.

Consequently, we may assume that R satisfies any preassigned *finite* set of identities of $\pi \times \pi$ matrices. Finally, an argument mimicing the argument appearing in [**Ra**, Theorem 3] enables us to settle this case.

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