BOOK REVIEWS

The decision problem: Solvable classes of quantificational formulas, by Burton
  Dreben and Warren D. Goldfarb, Addison-Wesley, Reading, Mass., 1979,
  xii + 271 pp., $27.50.

Unsolvable classes of quantificational formulas, by Harry R. Lewis, Addison-
  Wesley, Reading, Mass., 1979, xv + 198 pp., $13.50.

1. What is the decision problem in question? The original problem was to
find an algorithm deciding logical validity of sentences of the usual first-order
logic.

To illustrate the strength of the desired decision procedure let us show that it
would decide Fermat's Great Theorem. Choose a finitely axiomatizable first-
order arithmetic or set theory T such that (i) you believe that the axioms of T
are true and (ii) T is strong enough for the argument below. (The Bernays-Gödel
set theory would do for most mathematicians.) Let A be the conjunction of the
axioms of T, and let F be a sentence saying in the language of T that there is a
quadruple a, b, c, n of positive integers with $a^n + b^n = c^n$ and $n > 2$. If $A \to F$
is logically valid then, by (i), F is true and Fermat's Great Theorem fails. If
Fermat's Great Theorem fails and a, b, c, n form a failing quadruple then T
proves $a^n + b^n = c^n$ and $n > 2$; hence T proves F i.e. $A \to F$ is logically valid.

Historically it appeared more convenient to speak about satisfiability than
logical validity. A sentence S is called satisfiable (respectively finitely satisfi-
able) if there is a model (respectively a finite model) satisfying S. A sentence S
is satisfiable iff NOT S is not logically valid, S is logically valid iff NOT S is
not satisfiable. Thus decision problems for logical validity and satisfiability are
reducible to each other.

In order to give a feeling of the field let me quote some early results. In 1915
Löwenheim gave a decision procedure for satisfiability of monadic sentences—
those using only one-place predicates. He proved also that dyadic sentences—
those using only two-place predicates—form a reduction class for satisfiability.
(A class C of sentences is called a reduction class for satisfiability if there is an
algorithm assigning a sentence S' in C to any first-order sentence S in such a
way that S' is satisfiable if and only if S is so. Reduction classes for finite
satisfiability are defined similarly.) In 1931 Herbrand sharpened the latter
result showing that just three dyadic predicates suffice to get a reduction class
for satisfiability. In 1936 Kalmar achieved the result that even one dyadic
predicate suffices.

Recall that every first-order sentence can be effectively rewritten in a
so-called prenex form: with quantifiers in front. E.g.

$$\forall x(\exists y R(x, y) \quad \text{and} \quad \exists z R(z, x))$$

is equivalent to

$$\forall x \exists y \exists z (R(x, y) \quad \text{and} \quad R(z, x)).$$

In 1928 Bernays and Schönfinkel gave a decision procedure for satisfiability of $\exists^*\forall^*$ sentences i.e. prenex sentences of the form $\exists \cdots \exists \forall \cdots \forall$. Earlier, in 1920, Skolem showed that $\forall^*\exists^*$ sentences form a reduction class for satisfiability.

In 1928 Ackermann gave a decision procedure for satisfiability of $\exists^*\forall\exists^*$ sentences. Gödel, Kalmar, and Shütte, separately in 1932, 1933, and 1934, respectively, published decision procedures for satisfiability of $\exists^*\forall^2\exists^*$ sentences. Gödel proved also that $\forall^3\exists^*$ sentences form a reduction class for satisfiability.

All the above-mentioned classes which are decidable for satisfiability are decidable for finite satisfiability. All above-mentioned reduction classes for satisfiability are reduction classes for finite satisfiability.

Church in 1936 and Turing in 1937 solved negatively the original problem: there is no algorithm (at least no recursive procedure) deciding satisfiability of arbitrary first-order sentences. Trakhtenbrot in 1950 showed that no recursive procedure decides finite satisfiability of arbitrary first-order sentences. The decision problem (or the classical decision problem) became the problem of which classes of first-order sentences are decidable for satisfiability and/or finite satisfiability.

2. Different programs. A question arises as to which classes of first-order sentences are to be considered. One cannot go through all the continuum many classes. Also some classes are none of the pure logicians' business, e.g., classes of sentences $A \to S$ where A is the conjunction of the axioms of groups (or fields, etc.). Thus a direction, a program is needed.

Historically, central attention was given to classes of prenex sentences with restrictions on quantifier prefix and similarity type. (The similarity type of a sentence S is defined by the numbers of monadic, dyadic, etc., predicate symbols in S.) In a monograph of 1959 Surányi summarized the huge work done on undecidable prefix-similarity classes. In 1962 Buchi found a simple way to describe a Turing machine by an $\forall\exists\forall$ sentence in conjunction with an $\exists$ sentence. Almost immediately after that Kahr, Moore, and Want proved that $\forall\exists\forall$ case is undecidable (both for satisfiability and finite satisfiability) and completed the solution of the decision problem for prefix classes without equality.

The complete solution for the decision problem for prefix-similarity classes without equality was achieved in 1966. (I was lucky to get the final result.) If Gödel's claim of 1933—every satisfiable $\forall^2\exists^*$ sentence with equality is finitely satisfiable—is true, then the decision problem for prefix-similarity classes with equality has exactly the same solution. However Goldfarb proved recently that there is no primitively recursive decision procedure for satisfiability of $\forall^2\exists^*$ sentences.

Until now we considered sentences without function symbols. The decision problem for prefix-similarity classes of sentences with at least one function

symbol was completely solved too. The most complicated positive case was solved by Shelah in 1977.

In 1931 Herbrand gave a decision procedure for prenex sentences S such that the quantifier-free part of S is a conjunction of atomic formulas and negations of atomic formulas. In 1970 Krom proved that prenex sentences, whose quantifier-free parts are in conjunctive normal form with at most 2 disjuncts per conjunct, form an undecidable class. The decision problem for prefix-similarity subclasses of the Krom class is almost solved.

A prenex sentence is called Horn (after Alfred Horn) if its quantifier-free part is in conjunctive normal form with at most one negated disjunct per conjunct. The decision problem for prefix-similarity classes of Horn sentences is almost solved.

There were more programs around. Ackermann, Church, and Surányi surveyed the classical decision problem in the 1950's. Since then enormous progress has been made. But no book on the decision problem was published until 1979 (when the reviewed books appeared), and an important reason for that was that for many years the reviewed books were announced to survey the field and to appear soon.

3. The reviewed books. In the introduction to their book Dreben and Goldfarb write

> "In this book we treat questions of solvability by examining the Herbrand expansions of quantificational formulas.... The study of solvable classes has been bedeviled since its inception by overly syntactic and ad hoc arguments. It is our hope, however, to elicit the mathematical structures that underlie solvability. Our intent is not merely to compile a list of solvable and unsolvable classes. Rather, we aim at illuminating those features of expansions that give rise to decision procedures."

This is another kind of program. The authors realize it on about 270 pages of a very carefully written text. Terms are well defined. The Herbrand expansions technique is gradually developed and well demonstrated. It is an excellent (and certainly the best) text on the Herbrand expansions technique. But it does not live up to the declared intentions to "provide for the first time a unified treatment of the positive results of the decision problem," to give a systematic account of decidable classes. The authors never question their implicit premise that the Herbrand expansions technique is all-powerful, and never consider possible limitations of the technique. My criticism is

(i) the most important classical proofs were not simplified by the unified treatment,

(ii) there is no convincing evidence that the Herbrand expansions technique inspired many important new results, and

(iii) the book does not cover or even mention some of the most important results and programs.

Let me illustrate (i). In the monadic case, mentioned above, the usual (for model theorists) quantifier elimination works easier than the Herbrand expansions technique. If you are interested in a decision procedure for $\exists^*\forall^2\exists^*$ sentences without equality you will find the original Gödel paper much easier to read than Dreben-Goldfarb's book. (By the way Gödel's proof was simplified recently in a forthcoming Gurevich-Shelah paper. The simplification is achieved by a new method using random models and probabilities.) True, decidability of what the authors call the Maslov case is proved pretty neatly in the book, and one certainly may feel that Maslov's original method—called the inverse method—is overly syntactic. However Maslov proved a much stronger result involving function symbols.

Let me comment on (ii). Herbrand expansions are indeed relevant when you consider satisfiability of first-order sentences. The important paper of Buchi, mentioned in §2, makes an implicit use of Herbrand expansions. In this broad sense almost any paper on the classical decision problem uses Herbrand expansions. ("Good heavens! For more than 40 years I have been speaking prose without knowing it"—Molière.) I do not see, however, convincing evidence that the specific Herbrand expansions technique, developed in the book, inspired many important new results. It might motivate Goldfarb's impressive result, mentioned in §2, but his proof, published lately in a separate paper, does not use the technique of the book at all.

Let me exemplify (iii). The authors refuse to consider function symbols. Thus a most important and difficult part of the work on the classical decision problem is not even touched by the unifying treatment. Moreover, the analysis in the book seems to be not very relevant to the difficult cases with function symbols. The bibliography of the book does not reflect a big part of the work on the decision problem done in the last 20 years.

To summarize, it is a clever book developing a very useful and important technique. It is not a good survey of positive results on the classical decision problem.

It is much easier to review Lewis's book. The book is a clear one. A third of it is devoted to combinatorial systems useful to get undecidability results. It is an invaluable source for future work on the classical decision problem. In the rest of the book Lewis proves all the strongest undecidability results on prefix-similarity classes without equality or function symbols, and he proves all the strongest undecidability results, known by 1979, on prefix-similarity-Krom classes, on prefix-similarity-Horn classes, and on a couple of other programs. He mentions explicitly programs not covered in the book. He uses the Herbrand expansions technique. It is more convenient in some cases and less convenient in others. It is usually a clever trick which solves a hard undecidability problem. The language used (Herbrand expansions or whatever) should expose, not cloud, the trick. In spite of all this Lewis's exposition is excellent.

It is a pity that Lewis chose not to consider finite satisfiability which makes all the difference for theoretical computer science. Since the original papers usually did care about finite satisfiability, much stronger results could be obtained by modest additional effort.

4. Epilogue. "We view this book"—write Dreben and Goldfarb—

> "as a prologomenon to an abstract study of solvability and
> related notions, a study not concerned with particular classes.
> Is there an informative general criterion that distinguishes
> those syntactic restrictions that do from those syntactic re-
> strictions that do not lead to solvable classes? We hope our
> examination of the structural properties of expansions pro-
> vides the data and tools needed to attack this question, and
> points to the general concepts in terms of which an answer
> might be formulated."

It seems pretty clear to me that the desired informative general criterion
does not exist, and that this negative answer can be proved when the question
is formalized in a reasonable way. An old theorem of Tarski says for example
that $\{S$: the set of logically valid implications $S \to S'$ is decidable$\}$ is undecid-
able.

I do not know how bright the future of the classical decision problem is. It
seems clear to me however that the wealth accumulated during years of
research should be properly exposed. A comprehensive account of the decision
problem for prefix-similarity classes still needs to be written. It should treat
satisfiability and finite satisfiability, cases with or without equality, cases with
or without function symbols. Other most important programs should be
comprehensibly presented. Complexity of the decision procedures should be
treated, and connections between the classical decision problem and other
subjects (computer science in particular) should be shown.

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Pseudodifferential operators, by Michael E. Taylor, Princeton Univ. Press,
Princeton, N. J., 1981, 451 pp., $35.00.

The theory of partial differential equations, even only the linear ones, is one
of the vastest in mathematics. It has a great variety of subjects to study and an
even greater variety of tools, ranging from very general principles to really
special tricks. Most authors of books on this subject therefore deal with only a
rather limited number of topics. If well chosen these can be representative for a
reasonable part of the theory as a whole. In particular if it is the intention to
write a textbook then this is a wise strategy. To aim at greater completeness
requires not only a broad and deep knowledge of the field but also a heroic
stamina.

The book of Taylor belongs to the unusual class where completeness has
been a major goal. It treats almost all the important tools of linear partial