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PROPER HOLOMORPHIC MAPPINGS EXTEND SMOOTHLY TO THE BOUNDARY

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Biholomorphic mappings between smooth bounded domains in $\mathbf{C}^n$ are known to extend smoothly to the boundary in a wide variety of cases [7, 5, 1]. Much less is known about the boundary behavior of proper holomorphic mappings. In this communication, we sketch the proof of

THEOREM 1. *Suppose $f: D_1 \longrightarrow D_2$ is a proper holomorphic mapping between smooth bounded pseudoconvex domains contained in $\mathbf{C}^n$. If the Bergman projection associated to D_1 maps $C^\infty(\overline{D}_1)$ into $C^\infty(\overline{D}_1)$, then f extends smoothly to $\overline{D}_1$.*

Kohn has proved that the Bergman projection associated to a smooth bounded domain D maps $C^\infty(\overline{D})$ into $C^\infty(\overline{D})$ when D is strictly pseudoconvex [11], and more generally, when the boundary of D satisfies certain geometric conditions [12]. Diederich and Fornaess [6] have shown that these conditions are satisfied when the boundary of D is real analytic and pseudoconvex.

REMARKS. (A) Our proof of Theorem 1 uses arguments similar to those used in [2] where it was assumed that the Bergman projection preserved the space of functions which are real analytic up to the boundary. The additional complications encountered in the present work stem from the fact that the ring of germs of smooth functions is not a unique factorization domain.

(B) K. Diederich and J. E. Fornaess have informed us that they also have obtained a proof of Theorem 1 [8].

SKETCH OF THE PROOF OF THEOREM 1. A complete proof of this theorem will appear in [4]. In [3], it is shown that under the hypotheses of Theorem 1, the Jacobian determinant of f, $u = \mathrm{Det}[f']$, extends smoothly to $\overline{D}_1$ and uf^α extends smoothly to $\overline{D}_1$ for each multi-index α. Hence, we are faced with a division problem: to show that u divides uf in $C^\infty(\overline{D}_1)$, given that u and uf^α are in $C^\infty(\overline{D}_1)$ for each α. A necessary first step in attempting to solve this division problem is

LEMMA 1. *Under the hypotheses of Theorem 1, the Jacobian $u = \mathrm{Det}[f']$ vanishes to at most finite order at each boundary point of D_1.*

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We shall describe the proof of Lemma 1 at the end of this note.

The main tool used to prove Theorem 1 is an adapted version of the Mather division theorem. Let U be the upper half plane $\{\text{Im } z > 0\}$ in $\mathbf{C}$, and let $\overline{U}$ be its closure. Let Γ^N denote the set of functions $h(z, x)$ which are defined in a neighborhood of $(0, 0)$ in $\overline{U} \times \mathbf{R}^N$, which for fixed x are holomorphic in z on U, and which are smooth up to the boundary of $U \times \mathbf{R}^N$ near $(0, 0)$. Suppose that $F(z, x)$ and $G(z, x)$ are in Γ^N and that $(\partial^m F/\partial z^m)(0, 0) \neq 0$. Then there exist a neighborhood W of the origin in $\mathbf{C} \times \mathbf{R}^N$, a function $Q(z, x)$ in Γ^N, and smooth functions $a_k(x)$ such that for (z, x) in $W \cap \overline{U} \times \mathbf{R}^N$,

$$G(z, x) = Q(z, x) F(z, x) + \sum_{k=0}^{m-1} a_k(x) z^k.$$

The proof of this division theorem is a straightforward modification of the proof of the Mather division theorem given in, for example, **[10]**. In the usual way, an analogous version of the Malgrange preparation theorem follows for the class Γ^N.

We now make a change of variables to place the functions u and uf^α in the space Γ^{2n-2}. Let z_0 be a point in the boundary of D_1 such that $u(z_0) = 0$, and let $(z_1, z_2, \ldots, z_n)$ be holomorphic coordinates in a neighborhood of z_0 such that

(i) the z_1 direction is transverse to the boundary of D_1 at z_0, and

(ii) u vanishes to finite order m at z_0 in the z_1 direction.

Let $x \in \mathbf{R}^{2n-2}$ be given by $x = (x_2, y_2, \ldots, x_n, y_n)$ where $z_k = x_k + iy_k$. For fixed x, we may apply the Riemann mapping theorem in the variable z_1 to flatten out the boundary. Furthermore, this can be done smoothly in x. Under this change of variables, a holomorphic function on D_1 in $C^\infty(\overline{D}_1)$ is transformed into a function in the class Γ^{2n-2}.

Let f_1 denote the first component of the mapping f. The functions u and uf_1^j are in Γ^{2n-2} for $j = 1, 2, 3, \ldots$. We now apply the division theorem to $G = uf_1$ and $F = u$. We obtain

$$uf_1 = qu + r$$

where r is a polynomial in $z = z_1$ of order $m - 1$ with smooth coefficients in x. Note that $f_1 = q + r/u$. A simple induction argument using the fact that uf_1^j is in Γ^{2n-2} for each positive integer j reveals that r^{j+1}/u^j is in Γ^{2n-2} for all j.

We can assume that

$$u(z, x) = h(z, x) \left(z^m + \sum_{k=1}^{m-1} b_k(x) z^k \right)$$

where $h(0, 0) \neq 0$ and $h \in \Gamma^{2n-2}$ and the b_k's are smooth. Let $v(z, x) = z^m + \sum_{k=1}^{m-1} b_k(x) z^k$. At this point, we want to show that the derivatives

$(\partial^\nu/\partial z^\nu)(r/v)(z, x)$ are uniformly bounded by a constant C_ν which is independent of x for x in a neighborhood of 0. This is accomplished via

LEMMA 2. *Let $U_R = \{z \in U\colon |z| \leq R\}$ and let $r_k(z)$ and $v_k(z)$ be sequences of polynomials of the form*

$$r_k(z) = \sum_{j=0}^{m-1} a_{j,k} z^j \quad \textit{and} \quad v_k(z) = z^m + \sum_{j=0}^{m-1} b_{j,k} z^j,$$

where the coefficients $a_{j,k}$ and $b_{j,k}$ are bounded in absolute value by a constant M for all j and $k \geq 0$. Assume also that for each positive integer N, there is a constant $C_N > 0$ such that for all $k > 0$,

$$\operatorname*{Sup}_{z\in U_R} \left| D^N \frac{r_k^{m+1}}{v_k^m}(z) \right| \leq C_N.$$

Then there are constants C'_N such that for all k,

$$\operatorname*{Sup}_{z\in U_{R/2}} \left| D^N \frac{r_k}{v_k}(z) \right| \leq C'_N.$$

Using the lemma, one immediately obtains that the derivatives of $f_1(z, x)$ in the z variable are bounded in a neighborhood of $(0, 0)$ in $U \times \mathbf{R}^{2n-2}$. In the original coordinates, we have that the derivatives

$$\left| \frac{\partial^N}{\partial z_1^N} f_1(z_1, z_2, \ldots, z_n) \right|$$

are bounded near z_0 independent of $(z_1, \ldots, z_n)$.

Since there is a dense open subset of complex directions z_1 for which the above procedure can be carried out, it follows easily that f_1 is smooth up to the boundary near z_0. All the other components of f are treated analogously.

PROOF OF LEMMA 1. The classical Remmert proper mapping theorem states that f is a branched cover of some finite order m. Let $F_1, F_2, \ldots, F_m$ denote the m inverses to f which can be defined locally on D_2 away from the image of the branch locus of f. In [3], it is shown that any symmetric function of $F_1, F_2, \ldots, F_m$ extends smoothly to $\bar{D}_2$. Hence, the n functions defined on $\mathbf{C}^n \times D_2$ given by

$$P_i(z, w) = \prod_{k=1}^{m} (z_i - F_k(w)_i)$$

are in $C^\infty(\mathbf{C}^n \times \bar{D}_2)$. Note that if $w = f(z)$, then $P_i(z, w) = 0$ for $i = 1, 2, \ldots, n$.

Let z_0 be a boundary point of D_1 and let B_δ denote the ball of radius δ about z_0. Using the polynomials P_i and the fact proved in Range [13] and

Fornaess [9] that there is a positive integer η such that

$$\operatorname{dist}(z, bD_1)^\eta \leq \operatorname{dist}(f(z), bD_2) \leq \operatorname{dist}(z, bD_1)^{1/\eta},$$

it can be shown that the volume $B_\delta \cap D_1$ is mapped under f onto a volume whose measure is greater than a constant C times δ^M where M and C are independent of δ. Hence, using the fact that $|u|^2$ is equal to the real Jacobian determinant of f, we obtain that

$$\int_{B_\delta \cap D_1} |u|^2 \geq C\delta^M.$$

Therefore,

$$\operatorname*{Sup}_{B_\delta \cap D_1} |u|^2 \geq C'\delta^{M'},$$

for some constants C' and M' which are independent of δ. Hence u vanishes to at most finite order at z_0.

REMARK. It is interesting to note that the assumption that the domains D_1 and D_2 are pseudoconvex is only used at one point in the proof of Theorem 1. Pseudoconvexity is only assumed in order to obtain the crucial inequality

$$\operatorname{dist}(z, bD_1)^\eta \leq \operatorname{dist}(f(z), bD_2) \leq \operatorname{dist}(z, bD_1)^{1/\eta}.$$

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