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A SOLUTION TO A PROBLEM OF J. R. RINGROSE[1]

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We announce a solution to a multiplicity problem for nests posed by J. R. Ringrose approximately twenty years ago. This also answers a question posed by R. V. Kadison and I. M. Singer, and independently by I. Gohberg and M. Krein concerning the invariant subspace lattice of a compact operator. The key to the proof is a result concerning compact perturbations of nest algebras which was recently obtained by Niels Andersen in his doctoral dissertation. The complete proof of the general result as well as of a number of related results will appear elsewhere. A proof for the special case which answers Ringrose's original question is included herein.

Let H be infinite dimensional separable Hilbert space. A *nest* $\mathcal{N}$ is a family of closed subspaces of H linearly ordered by inclusion. $\mathcal{N}$ is *complete* if it contains $\{0\}$ and H and contains the intersection and the join (closed linear span) of each subfamily. The corresponding *nest algebra* alg $\mathcal{N}$ is the algebra of all operators in $L(H)$ which leave every member of $\mathcal{N}$ invariant. The *core* $\mathcal{C}_{\mathcal{N}}$ is the von Neumann algebra generated by the projections on the members of $\mathcal{N}$, and the *diagonal* $\mathcal{D}_{\mathcal{N}}$ is the von Neumann algebra (alg $\mathcal{N}$) $\cap$ (alg $\mathcal{N}$)*. $\mathcal{N}$ is *continuous* if no member of $\mathcal{N}$ has an immediate predecessor or immediate successor. Equivalently, $\mathcal{N}$ is continuous if the core $\mathcal{C}_{\mathcal{N}}$ is a nonatomic von Neumann algebra. $\mathcal{N}$ has *multiplicity one* (is multiplicity free) if $\mathcal{D}_{\mathcal{N}}$ is abelian, or equivalently, if $\mathcal{C}_{\mathcal{N}}$ is a m.a.s.a.

J. R. Ringrose posed the following question: Let $\mathcal{N}$ be a multiplicity free nest and $T: H \longrightarrow H$ a bounded invertible operator. Is the image nest $T\mathcal{N} = \{TN: N \in \mathcal{N}\}$ necessarily multiplicity free? Note that $T(\text{alg } \mathcal{N})T^{-1} = \text{alg}(T\mathcal{N})$, so it is natural to say that $T\mathcal{N}$ is the similarity transform of $\mathcal{N}$. Is multiplicity preserved under similarity? We show that the answer is no. It should be noted that a negative answer was conjectured in recent years by several mathematicians including J. Ringrose and W. Arveson.

The following key result is due to N. Andersen [1]. Let $\mathcal{LC}$ denote the compact operators in $L(H)$.

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THEOREM (ANDERSEN). *If H is separable, and if $\mathcal{N}$, $\mathcal{M}$ are arbitrary continuous nests in H, then there exists a unitary operator U such that* alg $\mathcal{N} + \mathcal{LC}$ $= U(\text{alg } \mathcal{M} + \mathcal{LC})U^* = \text{alg}(U\mathcal{M}) + \mathcal{LC}$.

Next, we answer Ringrose's question.

THEOREM 1. *Let $\mathcal{N}$ be a continuous nest of multiplicity* 1. *Then there exists a positive invertible operator $T \in L(H)$ such that $T\mathcal{N} = \{TN: N \in \mathcal{N}\}$ fails to have multiplicity* 1.

PROOF. By Andersen's theorem there exists a continuous nest $\mathcal{M}$ not of multiplicity 1 such that alg $\mathcal{M} + \mathcal{LC} =$ alg $\mathcal{N} + \mathcal{LC}$. Since for any algebra $\mathcal{A}$ we have $\mathcal{A} + \mathcal{LC}/\mathcal{LC} \approx \mathcal{A}/\mathcal{A} \cap \mathcal{LC}$, the algebras alg $\mathcal{M}$/alg $\mathcal{M} \cap \mathcal{LC}$ and alg $\mathcal{N}$/alg $\mathcal{N} \cap \mathcal{LC}$ are algebraically isomorphic. The diagonal $\mathcal{D}_{\mathcal{M}} =$ alg $\mathcal{M} \cap (\text{alg } \mathcal{M})^*$ is a nonabelian von Neumann algebra so contains a nonzero partial isometry v with orthogonal initial and final spaces. Let $\widetilde{S} = v + v^* - vv^* - v^*v + I$ and $\widetilde{P} = vv^*$. Then $\widetilde{S}^2 = I$ and $\widetilde{P}\widetilde{S}\widetilde{P} = 0$. Since $\mathcal{M}$ is continuous $\mathcal{D}_{\mathcal{M}}$ contains no compacts, so $\widetilde{P}$ has infinite rank. Thus via the algebraic isomorphism between quotients it follows that alg $\mathcal{N}$/alg $\mathcal{N} \cap \mathcal{LC}$ contains elements $\hat{P}$, $\hat{S}$ with $\hat{P}^2 = \hat{P} \neq 0$, $\hat{S}^2 = I$, $\hat{P}\hat{S}\hat{P} = 0$.

Let A and B be elements of alg $\mathcal{N}$ whose images in the quotient are $\hat{P}$ and $\hat{S}$ respectively. Then $A^2 - A$, $B^2 - I$ and ABA are contained in alg $\mathcal{N} \cap \mathcal{LC}$, and this is contained in the Jacobson radical $\mathcal{R}_{\mathcal{N}}$ of alg $\mathcal{N}$ since $\mathcal{N}$ is continuous. So B is invertible in alg $\mathcal{N}$. Also, a well-known result [cf. 9, Theorem 2.3.9] states that an element of a Banach algebra which is idempotent modulo the radical is equal modulo the radical to an idempotent. So there exists an idempotent $P \in$ alg $\mathcal{N}$ with $A - P \in \mathcal{R}_{\mathcal{N}}$. We have $P \neq 0$ since otherwise A would be in $\mathcal{R}_{\mathcal{N}}$ and hence $\hat{P}$ above would be a quasinilpotent idempotent, hence 0.

We have $PBP \in \mathcal{R}_{\mathcal{N}}$. Set $B_1 = B - PBP$. Then B_1 is also invertible in alg $\mathcal{N}$, and $PB_1P = 0$. Now set

$$S = B_1P + PB_1^{-1}(I - P) - B_1PB_1^{-1}(I - P) + I - P.$$

We have $PSP = 0$, and it can be verified that $S^2 = I$. (Let α denote the sum of the first two terms, β the sum of the remaining terms, and compute $\beta^2 = \beta$, $\alpha\beta = \beta\alpha = 0$, $\alpha^2 = I - \beta$.)

Let $R = I - 2P$. Then $R^2 = I$, $S^2 = I$, $RS \neq SR$. Let G be the group generated by R, S. We have $SRS = I - 2SPS$, and $PSP = 0$, so $PSRS = P = SRSP$. Hence R commutes with SRS since P does. It easily follows that

$$G = \{I, S, R, RS, SR, SRS, RSR, SRSR\}.$$

So G is a finite noncommutative group contained in alg $\mathcal{N}$. Set $T = (\Sigma_{g\in G}\, g^*g)^{1/2}$. Then $TGT^{-1} = \{TgT^{-1}: g \in G\}$ is a noncommutative group of unitaries contained in the diagonal of alg($T\mathcal{N}$), and thus $T\mathcal{N}$ fails to have multiplicity 1. □

Theorem 1 serves to answer an open question concerning invariant subspace lattices of compact operators due to Kadison and Singer [5] and to Gohberg and Krein [4]. An operator is said to be *hyperintransitive* if its lattice of invariant subspaces contains a multiplicity one nest.

COROLLARY 2. *There exists a nonhyperintransitive compact operator.*

PROOF. Let V be the Volterra operator. Then Lat V is a continuous multiplicity one nest. Let $\mathcal{N} = \text{Lat } V$ and let T be an invertible operator such that $T\mathcal{N}$ does not have multiplicity one. Since $\text{Lat}(TVT^{-1}) = T\mathcal{N}$ and since $T\mathcal{N}$ is a maximal nest the similarity TVT^{-1} is not hyperintransitive. □

REMARK. It was known for a number of years that a negative resolution to the Ringrose problem would yield Corollary 2. I believe that this connection was first observed by J. Erdos, and it was first shown to me by W. Arveson.

We strengthen Theorem 1 as follows.

THEOREM 3. *Let $\mathcal{N}$ be a continuous nest of multiplicity one. Then given $\epsilon > 0$ there exists a positive invertible operator $T \in L(H)$ with $T - I$ compact and $\|T - I\| < \epsilon$ such that $T\mathcal{N} = \{TN: N \in \mathcal{N}\}$ fails to have multiplicity one.*

A nest has *purely atomic core* if its core is generated by its minimal projections. The following shows that similarity transforms can fail to act "absolutely continuously" on nests.

THEOREM 4. *If $\mathcal{N}$ is a complete uncountable nest with purely atomic core there exists a positive invertible operator T such that $T\mathcal{N}$ does not have purely atomic core.*

A nest $\mathcal{N}$ is said to have the *factorization property* if every invertible positive operator T factors $T = A^*A$ for $A \in (\text{alg } \mathcal{N}) \cap (\text{alg } \mathcal{N})^{-1}$. Arveson [2] proved that nests of the "simplest type" have the factorization property. We generalize this to countable complete nests, and then show that these are the only ones with this property.

THEOREM 5. *A complete nest has the factorization property if and only if it is countable.*

In contrast, if we drop the requirement that A^{-1} also be in alg $\mathcal{N}$ we obtain.

THEOREM 6. *Let $\mathcal{N}$ be an arbitrary nest. Then every invertible positive operator T factors $T = A^*A$ for $A \in \text{alg } \mathcal{N}$, A invertible in $L(H)$.*

The following answers a question of J. Erdos [3].

THEOREM 7. *Let $\mathcal{N}$ be a continuous nest. Then the commutator ideal of* alg $\mathcal{N}$ *is not proper.*

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