BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 3, May 1982

ÉTALE K-THEORY AND ARITHMETIC

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The purpose of this note is to announce some new results about the algebraic K-theory of rings of integers in global fields.

THEOREM 1. *Let $\mathcal{O}$ denote the ring of integers in a number field K (i.e. a finite extension field of the rational numbers* $\mathbf{Q}$) *and let l be an odd prime. Then there are natural surjective maps*

$$(1.1)\quad \mathrm{ch}_{i,k}: K_{2i-k}(\mathcal{O}) \otimes \mathbf{Z}_l \longrightarrow H^k(\mathcal{O}[1/l], \mathbf{Z}_l(i)), \qquad k = 1 \text{ or } 2,\ 2i - k > 1.$$

REMARK. The requirement that l be an odd prime can be dropped if K is totally imaginary.

The groups on the right of (1.1) are continuous l-adic étale cohomology groups. Recall that $\mathbf{Z}/l^\nu(1)$ denotes the sheaf of l^νth roots of unity, $\mathbf{Z}/l^\nu(i) = (\mathbf{Z}/l^\nu(1))^{\otimes i}$, and $\mathbf{Z}_l(i) = \varprojlim_\nu \mathbf{Z}/l^\nu(i)$. D. Quillen has conjectured the existence of *isomorphisms* of type (1.1). B. Harris and G. Segal [4] have shown that (1.1) is surjective on torsion if $k = 1$; C. Soulé [6] in many cases proved surjectivity for $k = 2$ with $i < l$.

The surjectivity of (1.1) together with A. Borel's computation of $K_*(\mathcal{O}) \otimes \mathbf{Q}$ [1] gives a new proof of the existence [7] of isomorphisms

$$(1.2)\qquad \mathrm{ch}_{i,k} \otimes \mathbf{Q}: K_{2i-k}(\mathcal{O}) \otimes \mathbf{Q}_l \xrightarrow{\sim} H^k(\mathcal{O}[1/l], \mathbf{Q}_l(i)).$$

In particular, Theorem 1 implies that $\mathrm{ch}_{i,1}$ detects "Borel classes" in $K_{2i-1}(\mathcal{O})$ (i.e. basis elements for $K_{2i-1}(\mathcal{O})$/torsion). This leads to the following corollary, which is consistent with long-standing conjectures about the algebraic K-theory with finite coefficients of the algebraic closure of $\mathbf{Q}$.

COROLLARY 2. *For any integers $i \geqslant 1$ and $\nu > 0$ there exists a finite solvable field extension K' of K with ring of integers $\mathcal{O}'$ such that the image of $K_{2i-1}(\mathcal{O})$/torsion in $K_{2i-1}(\mathcal{O}')$/torsion is divisible by l^ν.*

Conjectures by S. Lichtenbaum [5] and work by Lichtenbaum and others relate the values of the Dedekind zeta function of K at negative integers to the number of elements of finite order in the groups $H^k(\mathcal{O}[1/l], \mathbf{Z}_l(i))$. For example, combining (1) with known properties of Bernoulli numbers gives the new result that $K_{134}(\mathbf{Z})$ contains an element of order 37.

Received by the editors November 12, 1981 and, in revised form, January 4, 1982.
1980 *Mathematics Subject Classification*. Primary 18F25; Secondary 12A60, 55N15.
[1]Research partially supported by the N. S. F.

We have also proved a theorem analogous to Theorem 1 in the function field case.

THEOREM 3. *Let A be a ring of integers in a function field of characteristic p (i.e., a finite extension of $\mathbf{F}_p(t)$) and let l be a prime different from p. Then there are natural surjective maps*

$$\mathrm{ch}_{i,k}\colon K_{2i-k}(A) \longrightarrow H^k(A, \mathbf{Z}_l(i)), \qquad k = 1 \text{ or } 2,\ 2i - k > 0.$$

This was first proved by C. Soulé for $i < l$ [6].

R. Thomason has recently awakened interest in a type of K-theory which is obtained from ordinary algebraic K-theory with coefficients by imposing a periodicity [8]. It is known that in many cases the periodic analogue of map 1.1 is either a split epimorphism [2] or even an isomorphism [9].

The proofs of Theorems 1 and 3 use étale K-theory, a twisted generalized cohomology theory on the étale homotopy type of a noetherian ring (or scheme). We extend the theory developed in [3] for varieties over an algebraically closed field to the setting of schemes over $\mathbf{Z}[1/l]$. In particular, this gives the following.

THEOREM 4. *There are natural transformations of ring-valued functions*

$$\hat{\varphi}_*\colon K_*(\ ) \longrightarrow \hat{K}^{\mathrm{et}}_*(\ ),$$
$$\bar{\varphi}_*\colon K_*(\ , \mathbf{Z}/l^\nu) \longrightarrow K^{\mathrm{et}}_*(\ , \mathbf{Z}/l^\nu), \qquad l^\nu \neq 2,$$

defined on the category of noetherian $\mathbf{Z}[1/l]$-algebras. For any finite, étale extension $A \longrightarrow A'$ of noetherian $\mathbf{Z}[1/l]$ algebras, $\hat{\varphi}_*$ and $\bar{\varphi}_*$ commute with the transfer maps on algebraic and étale K-theory.

There are relationships between étale K-theory and étale cohomology given by spectral sequences of Atiyah-Hirzebruch type.

PROPOSITION 5. *For any noetherian $\mathbf{Z}[1/l]$-algebra A of finite* mod *l étale cohomological dimension, there exist natural "fringed" spectral sequences*

$$E_2^{p,q} = H^p(A, \mathbf{Z}_l(-q/2)) \Rightarrow \hat{K}^{\mathrm{et}}_{-p-q}(A),$$
$$E_2^{p,q} = H^p(A, \mathbf{Z}/l^\nu(-q/2)) \Rightarrow K^{\mathrm{et}}_{-p-q}(A, \mathbf{Z}/l^\nu)$$

where $\mathbf{Z}_l(-q/2) = 0 = \mathbf{Z}/l^\nu(-q/2)$ unless q is a nonpositive even integer.

These spectral sequences necessarily degenerate for rings A of $\mathbf{Z}/l$ cohomological dimension at most 2 (e.g. $\mathcal{O}[1/l]$ in Theorem 1 or A in Theorem 3). In view of this, Theorems 1 and 3 are implied by the following theorem.

THEOREM 6. *Let A denote either a ring of integers in a function field of characteristic $p \neq l$, or $\mathcal{O}[1/l]$ with $\mathcal{O}$ a ring of integers in a number field. In the* *second case, assume that the quotient field of A is totally imaginary if* $l = 2$. *Then for any* $\nu > 0$ ($\nu > 1$ *if* $l = 2$) *the natural maps*

$$\bar{\varphi}_*: K_j(A, \mathbf{Z}/l^\nu) \longrightarrow K_j^{\mathrm{et}}(A, \mathbf{Z}/l^\nu) \qquad (j > 1)$$

are surjective.

In the case in which A contains a primitive l^νth root of unity (denoted ζ_{l^ν}) the proof of Theorem 6 is not difficult because $K_*^{\mathrm{et}}(A, \mathbf{Z}/l^\nu)$ is then periodic of period 2. To prove Theorem 6 in general, we combine this surjectivity of $\bar{\varphi}_*$ for $A' = A[\zeta_{l^\nu}]$ with the following secondary transfer theorem.

THEOREM 7. *Let* A, l, ν *be as in Theorem* 6, *let* $A' = A[\zeta_{l^\nu}]$, *and let* $T \in \mathrm{Gal}(A', A)$ *be a generator. Then for any* $i > 0$ *there exists a natural commutative square*

$$\begin{array}{ccc} \mathrm{Ker}\{K_i(A', \mathbf{Z}/l^\nu) \xrightarrow{1-T} K_i(A', \mathbf{Z}/l^\nu)\} & \longrightarrow & \mathrm{coker}\{K_{i+1}(A', \mathbf{Z}/l^\nu) \xrightarrow{\mathrm{tr}} K_{i+1}(A, \mathbf{Z}/l^\nu)\} \\ \downarrow \bar{\varphi}_i & & \downarrow \bar{\varphi}_{i+1} \\ \mathrm{Ker}\{K_i^{\mathrm{et}}(A', \mathbf{Z}/l^\nu) \xrightarrow{1-T} K_i^{\mathrm{et}}(A', \mathbf{Z}/l^\nu)\} & \longrightarrow & \mathrm{coker}\{K_{i+1}^{\mathrm{et}}(A', \mathbf{Z}/l^\nu) \xrightarrow{\mathrm{tr}} K_{i+1}^{\mathrm{et}}(A, \mathbf{Z}/l^\nu)\} \end{array}$$

whose lower horizontal arrow is surjective, where tr *denotes the transfer map.*

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