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RESEARCH ANNOUNCEMENTS

ON K_3 AND K_4 OF THE INTEGERS MOD n

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Quillen [7] defines an algebraic K-functor from the category of associative rings to that of positively graded abelian groups, with $K_i(R) = \Pi_i(\mathrm{BGLR}^+)$ for $i \geqslant 1$. K_1 and K_2 correspond respectively to the 'classical' Bass and Milnor definitions. The K-images of finite fields and their algebraic closures were computed by Quillen in [8]. Since then, there has been only a handful of complete calculations of any of the higher K-groups (K_i for $i > 2$). Lee and Szczarba [4] showed that the Karoubi subgroup $Z/48$ of $K_3(Z)$ was the full group. Evens and Friedlander [3] computed $K_i(Z/p^2)$ and $K_i(\mathrm{F}_p[t]/(t^2))$ for $i < 5$ and prime p greater than 3. Snaith in [1] and, with Lluis, in [5], fully determined $K_3(\mathrm{F}_{p^m}[t]/(t^2))$ for $m \geqslant 1$ and prime p other than 3.

This note summarizes computations of the groups $K_3(Z/n)$, and $K_4(Z/p^k)$ for $k > 1$ and prime $p > 3$. These complete the recent partial results on $K_3(Z/4)$ by Snaith and on $K_3(Z/9)$ by Lluis, and extend the work of Evens and Friedlander. The theorem stated below is consistent with the Karoubi conjecture that for odd primes, $\mathrm{BGL}Z/p^{k+}$ is the homotopy fibre of the difference of Adams operations, $\Psi^{p^k} - \Psi^{p^{k-1}}$. However, Priddy [6] has disproved the conjecture in the cases $p > 3$ and $k = 2$.

I am most grateful to Victor Snaith for his supervision of the thesis in which these results originally appeared. Details of the proofs can also be found in [1].

THEOREM. *Take* $k > 1$ *and* $0 < i \leqslant 2$.

(a) $K_{2i-1}(Z/2^k) = Z/2^i \oplus Z/2^{i(k-2)} \oplus Z/(2^i - 1)$. $K_{2i-1}(Z/p^k) = Z/p^{i(k-1)} \oplus Z/(p^i - 1)$ *if p is an odd prime. For all primes, the map*

$$K_{2i-1}(Z/p^{k+1}) \longrightarrow K_{2i-1}(Z/p^k)$$

induced by reduction is the obvious surjection.

(b) *For prime* $p > 3$, $K_{2i}(Z/p^k) = 0$. $K_2(Z/3^k) = 0$. $K_2(Z/2^k) = Z/2$.

K_1 is due to Bass, K_2 to Milnor, Dennis Stein.

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$K_3(Z/p^k)$ is isomorphic to $H^4(\mathrm{St}Z/p^k; Z)$ where the special linear group $\mathrm{SL}Z/p^k$ coincides with $\mathrm{St}Z/p^k$ modulo $K_2(Z/p^k)$. For odd primes, $K_4(Z/p^k)$ is recovered from the homology of $\mathrm{SL}Z/p^k$ using the Serre spectral sequence related to the natural inclusion $\mathrm{BSL}Z/p^{k+} \longrightarrow K(K_3(Z/p^k), 3)$. Thus the bulk of the proof of the theorem consists of computing the low dimensional group cohomology of $\mathrm{SL}Z/p^k$. Stability results of Wagoner [9] and others mean that it suffices to work with $\mathrm{SL}_n Z/p^k$ for large n prime to the order of the group of units in Z/p^k. In fact, we will assume that n is large and $n \equiv 1 \bmod p$. Our method is based on recursive definition of group extensions and detailed comparison of the resulting Lyndon-Serre spectral sequences.

The key set of extensions are those induced by reduction,

$$E(k) \qquad G_n^k = \ker r_k \xrightarrow{i_k} \mathrm{SL}_n Z/p^k \xrightarrow{r_k} \mathrm{SL}_n Z/p.$$

The initial step in the recursive analysis is provided at $k = 2$ by the calculations of Snaith ($p = 2$), Lluis (odd primes) and Evens and Friedlander ($p > 3$) of the E_2^{**} terms in the associated spectral sequence with coefficients in Z or Z/p. (G_n^2 is isomorphic to $M_n^{\sim} Z/p$, the zero trace $n \times n$ matrices over Z/p.) A specific resolution-level differential formula is derived, then applied to the spectral sequence $H^*(\mathrm{SL}_n Z/2; H^*(M_n^{\sim} Z/2; Z/4)) \Rightarrow H^*(\mathrm{SL}_n Z/4; Z/4)$ to complete the determination of $H^4(\mathrm{SL}_n Z/4; Z/4)$ and thence of $K_3(Z/4)$. For the odd primes, in particular $p = 3$, spectral sequence pairings and the Charlap and Vasquez [2] differential formula are exploited in order to avoid resolution level calculations in the integral spectral sequences associated with $E(2)$. So for all primes p, $H^4(\mathrm{SL}_n Z/p^2; Z)$ is known.

For the recursive step, the modules $H^i(G_n^k; Z)$ for $i \leqslant 4$ and $k > 2$ need first to be estimated. This is done through the spectral sequences associated with the central group extensions

$$\hat{E}(k) \qquad M_n^{\sim} Z/p \rightarrowtail G_n^k \xrightarrow{\pi_k} G_n^{k-1}, \quad k > 2.$$

Initially take Z/p coefficients. Because the base group in $\hat{E}(3)$ is an elementary abelian p-group, it is straightforward to apply the Hochschild-Serre formula for the d_2 differential. In an identical calculation to that which would be used to determine $H^*(M_n^{\sim} Z/p^2; Z/p)$ from the equivalent filtration, the full graded module $H^*(G_n^3; Z/p)$ is obtained when p is odd. When $p = 2$, an ad hoc computation of desired E^{**} terms must be employed. For $k > 3$ the differential formula cannot be neatly expressed. However, the image of the $d_2^{0,1}$ differential can be shown to be precisely the cokernel of $\pi_{k-1}^*: H^2(G_n^{k-2}; Z/p) \longrightarrow H^2(G_n^{k-1}; Z/p)$ by comparing low dimensional terms in the spectral sequence associated with $\hat{E}(k)$ with those in the sequence $H^*(G_n^{k-2}; H^*(M_n^{\sim} Z/p^2; Z/p)) \Rightarrow H^*(G_n^k; Z/p)$. From this, an isomorphism with the $k = 3$ spectral sequence is obtained if p is odd. For $p = 2$, each of the $H^*(G_n^k; Z/p)$ is isomorphic as $\mathrm{SL}_n Z/p$-modules if $k > 3$, and as groups if $k \geq 3$.

The modules $H^i(G_n^k; Z)$, $i < 4$, are determined from the Z/p-results using the integral spectral sequence associated with $\hat{E}(k)$ and the fact that in this situation, $d_3(1 \otimes \beta) = \beta d_2$ (β the Bockstein $H^*(-; Z/p) \longrightarrow H^{*+1}(-; Z)$). These modules are expressed in terms of direct summands and quotients of $H^*(M_n^{\sim} Z/p; Z/p)$, in particular, summands which are the $(\pi_3 \cdots \pi_k)^*$-images of $H^*(G_n^2; Z)$. It is then easy to show that the groups $H^i(\mathrm{SL}_n Z/p; H^j(G_n^k; Z))$ are isomorphic under $(\pi_3 \cdots \pi_k)^*$ for each $k \geq 2$ in total degree less than 6, if $(i, j) \notin \{(0, 5), (1, 4), (2, 3), (0, 4)\}$. Naturality of spectral sequences therefore provides for an isomorphism: $\ker i_2^* \longrightarrow \ker i_k^*$ restricting from the map: $H^4(\mathrm{SL}_n Z/p^2; Z) \longrightarrow H^4(\mathrm{SL}_n Z/p^k; Z)$ induced by reduction.

To find im i_k^*, first reconsider the spectral sequences associated with $\hat{E}(k)$. The $\mathrm{SL}_n Z/p$-invariants in the E_∞^{**} terms of total degree 4 are determined by specifically examining the action of the differential on invariants in the E_2^{**} terms. The Wagoner-Milgram **[10]** result that $K_3^c(Z/p)$, defined as $\varprojlim_k \Pi_3(\mathrm{BGL}Z/p^{k+})$, contains a copy of the p-adic integers is interpreted to mean that the subgroup of invariants in $H^4(G_n^k; Z)$ becomes arbitrarily large with increasing k. By studying possible representatives in $p \cdot H^4(G_n^k; Z)$ for decreasing k it can be recursively shown that all E_∞^{**} invariants represent invariants in the full group. With the Z/p results, we find the invariants in $H^4(G_n^k; Z)$ to be $Z/p \oplus Z/p^{2(k-2)+1}$ if p is odd, or $Z/2 \oplus Z/2 \oplus Z/2^{2(k-2)}$ if $p = 2$. Further, π_{k+1}^* may be taken to be the zero map on the first summand, and multiplication by p^2 on the last (and to be an isomorphism between the second summands if $p = 2$).

Next, an injection: $H^4(\mathrm{SL}_n Z/p^k; Z) \longrightarrow H^4(\mathrm{SL}_n Z/p^{k+1}; Z)$ induced by reduction is established by recursively determining the E_2^{**} terms in the spectral sequences $H^*(\mathrm{SL}_n Z/p^k; H^*(M_n^{\sim} Z/p; Z)) \Rightarrow H^*(\mathrm{SL}_n Z/p^{k+1}; Z)$, then inspecting differentials. This together with the known action of π_k^* permits the determination of the image of i_k^*, $k \geq 2$; it is as shown in the following commutative exact diagram which has now been set up for odd primes p. (The case $p = 2$ is entirely analogous.)

$$\begin{array}{ccccc}
\ker i_k^* & \rightarrowtail & H^4(\mathrm{SL}_n Z/p^k; Z) & \twoheadrightarrow & \mathrm{im}\, i_k^* = Z/p^{2(k-2)+1} \\
\cong \uparrow & & \uparrow & & \uparrow p^{2(k-2)} \\
\ker i_2^* & \rightarrowtail & H^4(\mathrm{SL}_n Z/p^2; Z) & \twoheadrightarrow & \mathrm{im}\, i_2^* = Z/p.
\end{array}$$

Then information about $H^4(\mathrm{SL}_n Z/p^2; Z)$ suffices to determine fully $H^4(\mathrm{SL}_n Z/p^k; Z)$ for all $k > 2$.

Finally, that $H^5(\mathrm{SL}_n Z/p^k; Z) = 0$ when $p > 3$ follows from the natural isomorphisms of the universal coefficient sequences

$$H^4(\mathrm{SL}_n Z/p^k; Z) \otimes Z/p \rightarrowtail H^4(\mathrm{SL}_n Z/p^k; Z/p) \twoheadrightarrow \mathrm{Tor}(H^5(\mathrm{SL}_n Z/p^k; Z), Z/p)$$

with the corresponding sequences when $k = 2$. The groups

$$H^i(\mathrm{SL}_n Z/3; H^j(M_n^{\sim} Z/3; Z/3))$$

for $(i, j) = (0, 4)$ and $(2, 2)$ which are needed to obtain $H^4(\mathrm{SL}_n Z/9, Z/3)$ are not yet available.

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