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Hopf algebras, by Eiichi Abe, Cambridge Univ. Press, 1980, translated by Hisae Kinoshita and Hiroko Tanaka, xii + 284 pp., $39.50.

The Hopf algebras under consideration are not the graded coalgebras/Hopf algebras of algebraic topology. Rather these are the Hopf algebras whose study was motivated by such examples as group algebras, universal enveloping algebras of Lie algebras and representative functions on Lie groups and Lie algebras [**5**, **6**, **8**]. Indeed the emphasis in Abe's book is on Hopf algebras which are either commutative or cocommutative. About twelve years ago another book by the same title appeared [**14**]. Since the first *Hopf algebras* was published, some open questions in Hopf algebra theory have been answered, and coalgebras/Hopf algebras have enjoyed wide application. This book presents the answer to one of these questions—uniqueness of Hopf algebra integrals. It presents two important areas of Hopf algebra applications—to algebraic groups and to field theory. A book like Abe's *Hopf* *algebras* makes it easy to learn the subject and see some of the importance. When I was learning Hopf algebras one of the main sources was a preprint about graded Hopf algebras. Rumor had it that several revisions of the preprint were in circulation and that the earlier the version the more readable. I only saw the version which eventually was published and so cannot testify to the falsity of the rumor. I will save my additional historical perspectives and rude noises for the appropriate moments in the following chapter by chapter discussion of the book.

Chapter one is a mostly self-contained review of the algebra used in the rest of the book. It substantially overlaps the material taught in a first year graduate algebra course. The chapter starts with modules, exact sequences, tensor products and goes on to algebras including graded, filtered and Lie algebras. There is some structure theory and representation theory for associative algebras plus a quick excursion into Hochschild cohomology. The chapter ends with a section on commutative algebra. Up until this point all proofs are included. In the commutative algebra section proofs of some theorems are omitted. Reading the commutative algebra section I get the impression that in an earlier life—perhaps the Japanese edition in 1977 or class notes—the proofs were included. In any event adequate references are provided and it was not a bad decision to omit these proofs.

Chapter two begins the presentation of coalgebras, bialgebras, and Hopf algebras. The chapter begins with defintions, many examples, and develops *elementary* properties. As with most developments of coalgebra theory the author uses a certain amount of linear duality. The needed results are neatly presented in §(2.1) and expressed in the appropriate topological language. §(2.2) is devoted to the representative bialgebra of a semigroup. This example is of historical importance since impetus for the study of Hopf algebras and the Hopf-algebraic study of algebraic groups springs from work by Hochschild and Mostow on representative functions [**5**, **6**, **8**]. The representative bialgebra is used in the next section to give the coalgebra dual to an algebra and later in Chapter five in the study of field theory.

The next few sections of Chapter two are devoted to algebra-coalgebra duality. These are fundamental to the study of coalgebras. Besides presenting specific results these sections show the reader how to apply results and intuition about algebras to coalgebras. For example the reader is introduced to the coradical of a coalgebra. Abe's treatment of the coradical is quite nice.

Chapter two continues with *intermediate* coalgebra theory, giving the dual Wedderburn decomposition theorem. This illustrates another feature: the reader learns that coalgebras—even infinite-dimensional coalgebras—behave dually to *finite*-dimensional algebras. The rest of the chapter rounds out a course in *intermediate* coalgebra theory. The presentation is clear. The selection of topics is good and contains important results about coalgebras/bialgebras as well as results which are important for applications of coalgebras/bialgebras/Hopf algebras. The chapter ends with structure theory of hyperalgebras.

A hyperalgebra is the type of Hopf algebra whose dual is a (local) *formal group*. Abe gives complete coalgebra structure results for hyperalgebras whose duals are power series rings. This shows for example that in characteristic zero hyperalgebras are universal enveloping algebras of Lie algebras. Abe's treatment of positive characteristic introduces the reader to some of the techniques and problems which arise in the more detailed coalgebra study of hyperalgebras.

It is true of this chapter and of much of the book that what Abe presents is complete in itself and also provides a good basis and good motivation for further study. This makes me wish that more avenues of further study were cited. For example after a theorem telling when the antipode has order two, Abe might have mentioned that there has been work on the order of the antipode and cited [**9**, **16**]. A more extensive bibliography would enhance even further the value of the book for reference. The excellent index and Abe's clear style make the book a fine reference work.

To avoid a wrong impression about the bibliography let me add that Abe does provide references to source material and some papers and books for further study. This is much better than my book which sadly has no bibliography.

Chapter three begins with comodules and rational modules. Besides being of importance to pure coalgebra theory these are among the basic building blocks of the Hopf-algebraic approach to affine algebraic groups, which is presented in the next chapter. In this chapter Abe also presents *bimodules* which are what some authors have called *Hopf modules*. Abe proves the bimodule structure theorem and then applies it in two succeeding examples. The second example leads into the theory of integrals for Hopf algebras which is begun a few sections later. Before presenting integrals Abe presents a few more constructions which besides being important in Hopf algebra theory show the similarity of Hopf algebras to groups.

When the integral is presented in §(3.1) the third example shows that integration over a compact group is indeed one correct way to think of the integral. The Hopf algebra setting of Maschke's theorem shows that certain group results just beg to be generalized to Hopf algebras.

Abe presents Sullivan's thesis [**10**], uniqueness of integrals. Chapter three ends with an account of duality between groups and Hopf algebras. Some results are proved and some mentioned. This is a perfect way to end the chapter before *Applications to algebraic groups*.

There are three main ways to approach the study of affine algebraic groups. The *classical* approach is to study subgroups of $Gl(n, k)$ which are the zeros of polynomials. The *functorial* approach is to study group (scheme) valued functors. The *Hopf algebraic* approach is to study commutative Hopf algebras. All of these studies overlap, and books written from one viewpoint usually mention the other two. There are examples in the literature of algebraic groups, of pairs of papers—each written from a different point of view—where apparently the second author was not aware that the results were already known but phrased in terms of one of the other approaches. Sometimes a result obtained by one approach takes unduly long to become known and available to workers using other approaches. I believe [**15**] is such an example. Here Takeuchi shows that a commutative Hopf algebra is projective as a module over a Hopf subalgebra and gives conditions for freeness. In the functorial approach this has to do with the morphism from a group to a quotient group being projective or free.

Some results about affine algebraic groups are stated more easily, proved more easily, or seem more natural from one of the three viewpoints than the others. When studying affine algebraic groups and specializing in one of the three approaches it is useful to have at least some familiarity with the other two. Abe's Chapter four provides an excellent survey of the Hopf-algebraic approach. Proofs are included and the coverage is broad. Readers already familiar with other approaches to affine algebraic groups will recognize results they already know put into the Hopf algebra setting, and perhaps in some cases even gain further insight into the results. Readers not already familiar with affine algebraic groups will find this a good introductioin to the Hopf-algebraic approach and may wish to pursue the study further. Waterhouse's book **[17]** presents the Hopf-algebraic and functorial approach. Hochschild's book [7] presents the Hopf-algebraic and classical approach with special emphasis on Lie algebras and characteristic zero. Borel **[3]** gives the classical treatment and Demazure and Gabriel **[4]** take the functorial approach.

In the simplest form of the Hopf-algebraic approach to affine algebraic groups the group G makes its appearance as a group of algebra homomorphisms from a finitely-generated commutative Hopf algebra H to the algebraically closed ground field. In my book there is an expository account of affine algebraic groups from this point of view. I—perhaps Abe as well—may be guilty of momentarily letting the reader think that the interest is in G as an abstract group. It is quickly apparent in Abe's Chapter four that it is the properties of G arising from H which are of interest. As in the rest of the book there are many illustrative examples.

Some specific topics about affine algebraic groups covered in Chapter four are the Zariski Topology, morphisms between affine algebraic groups, subgroups, quotient groups, the connected component of the identity, solvable groups, unipotent groups, tori, the character group, representation theory, and the Lie algebra and hyperalgebra of an affine algebraic group. All this and more in sixty-six pages. Because the needed Hopf algebra theory has already been developed the treatment does not feel hurried.

The notion of *homomorphism* and *derivation* can be unified and generalized by the coalgebra concept of *measuring*. *Groups of automorphisms* and *Lie algebras of derivations* may be replaced by *measuring bialgebras*. This is the basis—with some variation—for the application of Hopf algebras to field theory presented in Abe's last chapter. Historically the Hopf-algebraic approach developed in an attempt to eliminate the separability requirement of Galois theory and yet preserve the classification of intermediate fields, the descent theory, and other features of the Galois theory. Preserving the descent theory and applying it was accomplished in **[2]**. The analogue of Galois cohomology is presented in **[11]**. An unforseen benefit of the Hopf algebra approach was the theory of modular field extensions given in **[12]**. This paper grew out of my investigation into the similarities between purely inseparable field extensions and the divided power coalgebra structure of hyperalgebras. Once the results were obtained the Hopf algebras were omitted to make the paper more generally accessible. However Hopf algebraists reading **[12]** and seeing the role of higher derivations may have guessed the truth.

Getting a *useful* classification of intermediate fields by the Hopf algebra approach has not been so easy. There have been variations on the measuring Hopf algebra theme. One variation for purely inseparable extensions is to use groups based on higher derivations. Another variation is to use K/k bialgebras as is done in Abe's fifth chapter. K/k bialgebras seem to have been developed simultaneously and independently by Winter **[18]** and myself **[13]**. From the start Winter's applications were to the theory of field extensions; mine were elsewhere. Abe's treatement of K/k bialgebras is based on Winter's work **[18]**.

A K/k bialgebra is a gadget H which is simultaneously a coalgebra over a field K and an algebra over the subfield k, where the algebra and coalgebra structures are suitably related. If Δ: $H \to {}_K H \otimes_K {}_K H$ is the diagonalization then—as for ordinary bialgebras—one wishes Δ to be an algebra map. The trouble is that ${}_K H \otimes_K {}_K H$ is not an algebra since H is only a k algebra and K may not be central in H. Winter and Abe get around this in a manner which relies on the elements chosen to represent tensors. Their approach leads to the question of whether a certain equality holding for one choice of elements implies that it holds for all choices of elements.

For the K/k bialgebras H which Winter and Abe apply to field theory, the diagonal map Δ always has image in a certain submodule of ${}_K H \otimes_K {}_K H$ which in [13] I called $H \times_K H$ and showed that $H \times_K H$ has a natural k algebra structure. In [13] part of the definition of K/k bialgebras—called $\times_K$ bialgebras there—was that Δ should be an algebra map from H to $H \times_K H$. Not only is this analogous to Δ being an algebra map for ordinary bialgebras, it eliminates the matter of elements chosen to represent tensors.

Using K/k bialgebras Abe presents a wide range of topics from field theory. These include the Jacobson-Bourbaki theorem, Jacobson's exponent one "Galois theory" using p Lie algebras, Winter's splitting group—toral bialgebra theory, and the theory of modular extensions.

The book ends with an appendix on categories and functors.

My *Hopf algebras* book was published in 1969. Abe's book shows the growth of the subject since then. His book is good for reference, good for self-study, and good as a graduate text for a second year or later course.

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A formal background to mathematics, by R. E. Edwards, Universitext, Springer-Verlag, New York-Berlin-Heidelberg, Volumes 1a and 1b, 1979, xxxiv + 933 pp., $29.80; Volumes 2a and 2b, 1980, xivii + 1170 pp., $39.80.

Before I can review this unusual book adequately, it is necessary to consider how academic mathematicians (such as readers of this Bulletin) have gradually come to expect mathematics books to be written in a standard way. About a century ago, for example, books on Physics were appearing in Britain, written by confident authors who declared in enthusiastic Prefaces that their readers could now understand the mysteries of the Universe, God's handiwork, etc. By the 1920's the style had changed and a different type of author would say in his Preface 'This book contains all that is necessary for the B.Sc. Examinations of . . . '. In Mathematics, as well as Physics and other disciplines, the Enthusiast still writes, but he usually conforms to the American euphemism of the 'Reward Structure'; his readership may not be working for B.Sc. degrees, but they are in a hurry and want him to get cracking. They want the gold nuggets of the mathematical mine, without the view from the mountain, or disquisitions upon the systems fed by the gold. Among other things, therefore, they are unlikely to dwell on his Preface (especially if it is long and complex), and they expect a flow of information that is organised and assembled in a conventional way, to get to the point as quickly as possible. Of course, there is also an unspoken convention as to what this 'point' is, and the convention is reinforced by almost every new book that