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J. H. B. KEMPERMAN

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An introduction to nonharmonic Fourier series, by Robert M. Young, Academic Press, New York, 1980, x + 246 pp., $32.00.

This is a book about a branch of a branch of analysis; a twig you might say. A useful twig I should add, and one bearing many fine blossoms.

The subject is full of neat results and satisfactory resolutions of open problems, and we'll be taking a look at some of these. At first there were nonharmonic sines and cosines, sets of the form $\{\sin \lambda_n x\}$ and $\{\cos \lambda_n x\}$ in which $\{\lambda_n\}$ is a set of real numbers. Their study was initiated by J. L. Walsh [**10**] at the suggestion of G. D. Birkhoff. Only with the appearance of Paley and Wiener's colloquium publication [**9**] in 1934 did nonharmonic Fourier (NHF) analysis really get under way. After having improved on a result stemming from O. Szász's answer to a problem of G. Pólya's about nonharmonic sines and cosines, they go on to "... discuss the closure of the set $\{e^{i\lambda_n x}, 1\}$ ...", i.e., the property that only the null member of $L^2(-\pi, \pi)$ is orthogonal to every member of the set (the word "closure" is not, thankfully, used any more for this property, having been superseded by "completeness"). Incidentally a little later we read "... the only discussion of a case where the sole restriction on λ_n ... is one of the form $|\lambda_n - n| < L < \infty$ is due to Wiener". This is rather misleading since the paper referred to is about $\{\cos \lambda_n x\}$, not about sets of complex exponentials $\{e^{i\lambda_n x}\}$.

Thus was our subject born, and it is astonishing how much later work has its origins in this seminal effort of Paley and Wiener (I tend to think of it as the "big bang" of NHF analysis). The problems center chiefly on completeness and basis properties of sets $\{e^{i\lambda_n x}\}$, and connections with ordinary Fourier series are made via "equi-convergence" theorems; but more of this anon.

In order to give some idea of how the subject has developed we can hardly do better than trace some of the strands of thought emanating from [9].

One of these strands involves the idea of *stability*. Nowadays we have the

PALEY-WIENER STABILITY THEOREM. *Let B be a Banach space with norm $\| \ \|$ and basis (φ_n). Then (ψ_n), another sequence in B, is also a basis if*

$$\left\| \sum a_n(\varphi_n - \psi_n) \right\| \leqslant \lambda \left\| \sum a_n \varphi_n \right\|$$

for some λ, $0 < \lambda < 1$, and all finite sequences (a_n) of scalars. Here a *basis* is a sequence (φ_n) in B such that each member of B is represented by a unique norm convergent expansion $\sum b_n \varphi_n$. In Paley and Wiener's original version B was $L^2(-\pi, \pi)$ and the conclusions were weaker; they used it to show that for real λ_n, $\{e^{i\lambda_n x}\}$ is complete in $L^2(-\pi, \pi)$ if

$$(*) \qquad |\lambda_n - n| \leqslant D < c; \; n = 0, \pm 1, \ldots,$$

with $c = \pi^{-2}$. They then raised the question whether the constant π^{-2} could be improved.

This kind of problem was soon to be studied intensively by N. Levinson [8]. One of his results was that the best possible value for c is $\frac{1}{4}$. In a more comprehensive result he showed that the completeness (suitably defined) holds in $L^p(-\pi, \pi)$, $1 < p < \infty$, if

$$|\lambda_n| \leqslant |n| + 1/2p,$$

$1/2p$ being best possible. Here λ_n can be complex.

It was not long before questions began being asked about whether $\{e^{i\lambda_n x}\}$ could satisfy the stronger property of being a basis for L^p. To this end R. J. Duffin and J. J. Eachus gave a method of exploiting the Paley-Wiener stability theorem by expanding $\varphi_n - \psi_n$ in a special way. By comparing $e^{i\lambda_n x}$ with e^{inx} via a power series expansion they showed that $\{e^{i\lambda_n x}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ if $(*)$ holds with $c = \pi^{-1} \log 2$. Here a Riesz basis (φ_n) is one which satisfies

$$A\left\{ \sum |a_k|^2 \right\}^{1/2} \leqslant \left\| \sum a_k \varphi_k \right\| \leqslant B\left\{ \sum |a_k|^2 \right\}^{1/2}$$

for constants A, B such that $0 < A \leqslant B < \infty$, and for all finite sequences (a_n) of scalars.

Now $\pi^{-1}\log 2$ is approximately 0.2206, so one wonders whether this last result admits any improvement. M. I. Kadec, using the Duffin-Eachus method without apparently realising it, replaced their power series expansion with a special trigonometrical one, and was able to get the result under $(*)$ with $c = \frac{1}{4}$, but only for real λ_n. His attempt to extend this to complex λ_n ran into trouble as the author has pointed out (p. 223). Young has shown that the result does hold under Kadec's condition

$$(**) \qquad |\mathrm{Re}\, \lambda_n - n| \leqslant D < c \text{ and } |\mathrm{Im}\, \lambda_n| \leqslant M,$$

with $c = \frac{1}{4}$. Even this result has by now yielded to further generalisation.

Another strand of thought emanating from [9] concerns a method which uses properties of entire functions. Suppose $g \in L^2(a, b)$ for example, and

$f(z)$ is entire of exponential type. Then so is $F(z) = \int_a^b f(zx)g(x)\,dx$. Further, $F(\lambda_n) = 0$ if $\int_a^b f(\lambda_n x)g(x)\,dx = 0$. Because of this the completeness of $\{f(\lambda_n x)\}$ is closely tied up with the nature of the zeros of $F(z)$. For example $F(\lambda_n) = 0$ for some set $\{\lambda_n\}$ may imply $F(z) \equiv 0$, and if further this implies that g is null then the L^2 completeness of $\{f(\lambda_n x)\}$ is guaranteed. Paley and Wiener exploited this idea in many ways and several early results are due to them. A standard result here, get-at-able via Carleman's theorem, is the following.

EXAMPLE. $\{e^{i\lambda_n x}, 0 < \lambda_1 < \lambda_2 < \dots\}$ is complete in $L(-a, a)$ if $\lim n/\lambda_n > a/\pi$, $a < \pi$.

Several important ideas are illustrated by this example. For instance completeness is asserted under a condition of *density*; a sequence (λ_n) has density d if $\lim n/|\lambda_n| = d$, where (λ_n) is arranged in nondecreasing order. There are many other versions of our example, featuring other types of density. A particularly strong one due to N. Levinson states that the completeness also holds if d is replaced with the Pólya maximum density.

Another idea illustrated by our example is that of the *completeness radius* for $\{e^{i\lambda_n x}\}$. This is defined to be the supremum of all numbers a such that $\{e^{i\lambda_n x}\}$ is complete in $C[-a, a]$; this quantity doesn't change if C is replaced with L^p. Our example asserts that the completeness radius of the given set is not less than πd. A. Beurling and P. Malliavin have discovered a density that allows a very comprehensive result, which says that the completeness radius of $\{e^{i\lambda_n x}\}$, λ_n complex, is always proportional to this special density.

Here we must also mention the notion of *excess* and *deficiency*. Let's notice first that we certainly need the whole trigonometrical set $\{e^{inx}, n = 0, \pm 1, \dots\}$ for completeness in $L(-\pi, \pi)$; but $\{e^{inx}, n = 1, 2, \dots\}$ is complete in $L(-a, a)$, $a < \pi$, by our example, so its completeness radius is π and it would exhibit a high degree of deficiency if viewed over $(-\pi, \pi)$. A set $\{\varphi_n\}$ is said to have deficiency k in B if k is the smallest number of terms which must be adjoined to it to make it complete in B (and a similar definition for excess). For example the trigonometric set is complete in $L^p(-\pi, \pi)$, $1 \leqslant p < \infty$, but has deficiency 1 in $C[-\pi, \pi]$. The terminology and several early results are due, as you will have guessed, to Paley and Wiener.

The last strand originating with Paley and Wiener that we shall discuss involves the idea of *equiconvergent series*. The idea seems to have been introduced, into the present theory at least, by Walsh (op. cit.), but the first results for NHF series are once again Paley and Wiener's. Two series $\Sigma\, a_n$ and $\Sigma\, b_n$ are said to be equiconvergent at a point if $\Sigma(a_n - b_n)$ converges to zero there. Walsh pointed out several consequences of equiconvergence; for example if Gibb's phenomenon occurs for one series it occurs for the other. It is clear that the convergence at a point for one series implies it for an equiconvergent one, so that equiconvergence with ordinary Fourier series automatically establishes Carleson-Hunt type results for NHF series. Indeed when is a NHF series equiconvergent with an ordinary Fourier series? It turns out that the classical results are really part of more comprehensive theorems about completeness, presence of unique biorthogonal sets, etc., the completeness parts of some of which we have already mentioned. Thus the equiconvergence part of Paley and Wiener's contribution was that the ordinary Fourier and the NHF series for $f \in L^2(-\pi, \pi)$ are unformly equiconvergent on compact subsets of $(-\pi, \pi)$ if $(*)$ holds with $c = \pi^{-2}$. Levinson generalised to L^p, $1 < p \leqslant 2$, under $(*)$ with $c = (p - 1)/2p$. An important 1971 result of A. M. Sedleckiĭ gives the result under $(**)$ with $c = (p - 1)/2p$. Sedleckiĭ has proved many other equiconvergence theorems during the past decade.

Open questions are always with us and the book mentions several of them. Let me quote two closely related ones. The first has been raised by the author, your reviewer and everyone else who has given the matter any thought.

Q1. Is every basis of complex exponentials for $L^2(-\pi, \pi)$ a Riesz basis?

An affirmative answer would, for example, take much of the sting out of results of the kind which assert the preservation of the Reisz property of bases under perturbations of (λ_n). A related question is

Q2. Is $\{e^{\pm i(n-1/4)x}, n = 1, 2, \ldots\}$ a basis for $L^2(-\pi, \pi)$?

It is known that the Riesz property fails for this set, hence an affirmative answer to **Q2** would imply a negative answer to **Q1**.

Now let's look at some applications of NHF series. Nonharmonic sine and cosine series, as well as NHF series have been used to solve various equations that arise in applied fields where ordinary Fourier series are inappropriate. For example there have been applications to diffusion processes (of growth stimulants through carrot roots), to control theory and to problems of the slowing down of neutrons.

Another kind of application is to the sampling theory of signals in electrical engineering. By Fourier transformation one maps e^{isx}, $x \in (-\pi, \pi)$, to $w(s, t) = [\sin \pi(s - t)]/\pi(s - t)$, $t \in \mathbf{R}$; under such a mapping, to $\{e^{inx}, n = 0, \pm 1, \ldots\}$ there corresponds $\{w(n, t), n = 0, \pm 1, \ldots\}$, the functions of Whittaker's *cardinal* series which is used by electrical engineers (often under the name "Shannon sampling series") for the interpolation, or sampling, of signals. It was G. H. Hardy **[5]** (a reference missed by the author) who pointed out that the orthogonality and completeness properties of $\{w(n, t)\}$ are consequences of the unitary character of the Fourier transform, and it was he who named their closed linear span the "Paley-Wiener space". Of course *the* Paley-Wiener theorem (in the form it took when it was young and still trailed clouds of glory) asserts that this space consists of the restrictions to $\mathbf{R}$ of entire functions of exponential type π which are also in $L^2(\mathbf{R})$; and equivalently, it consists of those members of $L^2(\mathbf{R})$ whose inverse Fourier transforms have support on $[-\pi, \pi]$. In the same way from $\{e^{i\lambda_n x}\}$ we get $\{w(\lambda_n, t)\}$, the functions of the "jittered" cardinal series (e.g., **[6, 11]**). Jitter refers to the way in which λ_n deviates from n and in engineering applications is usually stochastic in nature rather than deterministic. Although much work has been done on the stochastic nature of jittered sampling (e.g., **[2, 3]**), I have yet to see in print, and would very much like to, a direct study of $\{e^{i\lambda_n x}\}$ in which (λ_n) is regarded as a stochastic process.

I was disappointed to find almost no mention of applications in the book.

Some references are given but only to the first group I mentioned (on p. 210). Surely this subject would benefit from what I hope is the current trend towards a more healthy interaction than has been the case in the recent past between mathematics and its applications.

Perhaps applications will soon demand a multi-dimensional NHF theory. We know that this is a highly nontrivial matter for ordinary Fourier series (e.g., [**1**]). How would notions like equiconvergence, the completeness radius, etc., extend to higher dimensions? What should one make, for example, of the fact [**4**] that $\{e^{i\mathbf{n}\cdot\mathbf{x}}, \mathbf{n} = (n_1, n_2), \mathbf{x} = (x_1, x_2)\}$ fails to be a (spherical partial sum) basis for $L^p, p \neq 2$?

Only one more groan about the book. In the preface the author states that "much of the material appears in book form for the first time". This needs some qualification. Two introductory chapters, the first on bases in Banach spaces, the second on entire functions, account for half the book and the author couldn't have been referring to this material. Chapter three deals with completeness and Chapter four with moment problems and bases. A reference [**7**] missed by the author contains at least some material from each of the four chapters. And after all Paley and Wiener [**9**] and Levinson [**8**] are books too.

Of course there is material here in a book for the first time and the author has done an excellent job on it; a case in point is his précis (Chapter 4, §7) of Duffin and A. C. Schaeffer's work on frames.

This is an introductory book and the experienced analyst will find the pace slow. Even so, Young has managed to incorporate a great deal of information of which I have been able to survey only a small and largely classical selection. There is a useful bibliography of 260 items. The book is meticulously carefully written and laid out in a very friendly manner by its publishers. It deserves, and not merely because it is the first and only book devoted entirely to this field, to be on the bookshelf (no, on the desk top, open) of everyone who wants to know about this interesting topic.

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J. R. HIGGINS

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Polynomial identities in ring theory, by Louis Halle Rowen, Academic Press, New York, 1980, xx + 365 pp., $39.50.

Rings satisfying a polynomial identity (PI) occupy an important place in modern noncommutative ring theory. Louis Rowen's book is an up to date, thorough presentation of PI-theory and related topics.

PI-theory was launched by a paper of Kaplansky in 1948 and "born again" with the discovery, by Formanek in 1971, of central polynomials. For convenience in this review we shall assume that our rings are algebras over fields. Thus, if A is an algebra over the field F, we say that A is a PI-ring if there exists a nonzero polynomial $p(x_1, \ldots, x_t)$ in noncommuting indeterminates with coefficients in F such that $p(a_1, \ldots, a_t) = 0$ for all possible substitutions of elements $a_1, \ldots, a_t$ in A. We say that p is an identity for A and that A is a PI-ring of degree d if d is the least degree of a polynomial which is an identity for A.

Commutative rings, obviously, satisfy $x_1x_2 - x_2x_1$. Nilpotent rings satisfy x^N where N is the index of nilpotence of the ring. Subalgebras and factor algebras of PI-rings are also PI-rings.

Before proceeding to the most important class of PI-rings, we pause for a definition. The nth standard polynomial S_n is $\Sigma_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$ where the sum runs over the symmetric group, $\mathfrak{S}_n$, on n letters and the x's are noncommuting variables. A celebrated theorem of Amitsur and Levitzki asserts that S_{2n} is an identity for the $n \times n$ matrices over a commutative ring. It is not difficult to see that $2n$ is the least degree of a polynomial identity for $n \times n$ matrices. Therefore, factor algebras of subalgebras of matrices are PI-rings–though not all PI-rings arise this way.

An important theme in the theory of PI-rings is the study of the "closeness" of classes of PI-rings to matrices over commutative rings and finding "tight" connections between PI-rings and their centers. Kaplansky proved that a primitive PI-ring is simple and finite dimensional over its center which is a field. Amitsur later showed that PI-rings with no nonzero nilpotent ideals are embeddable in matrices over commutative rings. To put this last result in perspective we note an observation of P. M. Cohn: the exterior algebra on an