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SPECTRAL PROPERTIES OF SOME NONSELFADJOINT OPERATORS[1]

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ABSTRACT. Let A be a compact linear operator on a Hilbert space H, $s_n(A) = \{\lambda_n(A^*A)\}^{1/2}$, Q be a compact linear operator, $I + Q$ be invertible, $B = A(I + Q)$. We prove that $s_n(B)s_n^{-1}(A) \to 1$ as $n \to \infty$. If $|Qf| \leq c|Af|^a|f|^{1-a}$, $a > 0$, $c > 0$, $f \in H$ and $s_n(A) = c_1 n^{-r}\{1 + O(n^{-q})\}r$, $q > 0$, then $s_n(B) = s_n(A)\{1 + O(n^{-\gamma})\}$, where $\gamma = \min\{q, ra(1 + ra)^{-1}\}$. This estimate is close to sharp. We also give conditions sufficient for the root system of B to form a Riesz basis with brackets of H. Applications to elliptic boundary value problems are given.

1. Notations, definitions. Let H be a separable Hilbert space, A and Q be compact linear operators on H, $B = A(I + Q)$, $\lambda_n(A)$ be the eigenvalues of A, $s_n(A) = \lambda_n\{(A^*A)^{1/2}\} = \{\lambda_n(A^*A)\}^{1/2}$ be the s-values of A (singular values of A), c be various positive constants, $\mathbf{R}^d$ be the Euclidean d-dimensional space, $D \subset \mathbf{R}^d$ be a bounded domain with a smooth boundary, L be a positive definite in $L^2(D)$ elliptic operator of order l and M be a nonselfadjoint differential operator of order $m < l$. We define $s_n(L) = \{s_n(L^{-1})\}^{-1}$. Let $A\phi = \lambda\phi$, $\phi \neq 0$. With the pair (λ, ϕ) one associates the Jordan chain defined as follows: consider $(*)\ A\phi^{(1)} - \lambda\phi^{(1)} = \phi$. If this equation is not solvable then one says that there are no root vectors associated with the pair (λ, ϕ). If $(*)$ is solvable then consider the equations $(**)\ A\phi^{(j)} - \lambda\phi^{(j)} = \phi^{(j-1)}$, $j = 1, 2, \ldots, \phi^{(0)} \equiv \phi$. It is known [1], that if A is compact then there exists an integer N such that $(**)$ will not be solvable for $j > N$. In this case vectors $\phi^{(1)}, \ldots, \phi^{(N)}$ are called the root vectors associated with the pair (λ, ϕ), $(\phi, \phi^{(1)}, \ldots, \phi^{(N)})$ is called the Jordan chain associated with the pair (λ, ϕ). Consider the eigenvectors $\phi_1, \ldots, \phi_q$ corresponding to the eigenvalue λ and all the root vectors associated with the pairs (λ, ϕ_p), $p = 1, \ldots, q$. The linear span of the eigen and root vectors corresponding to λ is called the root space corresponding to λ. The collection of all eigen and root vectors of A is called its root system. Let us define Riesz's basis of H with brackets. Let $\{f_j\}$ be a linearly independent system of elements of H, $\{h_j\}$ be an orthonormal basis of H, and $m_1 < m_2 < \cdots < m_j \longrightarrow \infty$ be a

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sequence of integers. Let $H_j(F_j)$ be the linear span of vectors

$$h_{m_j}, h_{m_j+1}, \ldots, h_{m_{j+1}-1}, (f_{m_j}, \ldots, f_{m_{j+1}-1}),$$

T be a linear bounded invertible operator from H onto H, $TF_j = H_j$, $j = 1, 2, \ldots$. Then the system $\{f_j\}$ is called a Riesz basis of H with brackets. If $m_j = j$ then $\{f_j\}$ is called a Riesz basis of H. If a root system of A forms a Riesz basis of H with brackets then we write $A \in R_b(H)$. If it forms a Riesz basis then we write $A \in R(H)$. The range of A is denoted by $R(A)$ lim means lim as $n \longrightarrow \infty$, $N(A) = \operatorname{Ker} A = \{\phi: A\phi = 0\}$, $\{0\}$ denotes the set consisting of the zero element of H.

2. **Introduction.** Two questions will be discussed: (1) When is $s_n(B) \sim s_n(A)$ and what is the order of the remainder? (2) When does $B \in R_b(H)$? There are few known results connected with question (1). The results are due to H. Weyl, Ky Fan and M. G. Krein (see [2]), and the author [3]. It seems that there were no abstract results on the perturbations preserving asymptotics of spectrum with estimates of the remainder. In Theorem 1 (§3 below) such a result is given. In [2] there are some results about completeness of the root systems of certain operators. In Theorem 2 an abstract result which gives an answer to question (2) is given. In Theorem 3 some spectral properties of nonselfadjoint elliptic operators are presented. F. Browder [1, Chapter 14, Theorem 28] proved completeness of the root system of $L + M$ in $H = L^2(D)$. We prove that $L + M \in R_b(H)$ by applying Theorem 2. In order to do this note that $(L + M)^{-1} = A(I + Q)$, where $A = L^{-1}$, $Q = -(I + ML^{-1})^{-1}ML^{-1}$. During the last decade there was a great interest among physicists and engineers in question (2) and some results due to Markus, Kacnelson, Agranovich and others were used [4] (see also Appendix 10 in [3], [5], [6]).

3. **Results.** We will not repeat in this section the notations and assumptions of §1 but they are assumed to be valid.

THEOREM 1. *If $N(I + Q) = \{0\}$, $\dim R(A) = \infty$, then* $\lim s_n(B)s_n^{-1}(A) = 1$. *If $|Qf| \leqslant c|Af|^a|f|^{1-a}$, $a > 0$, for all $f \in H$ and $s_n(A) = cn^{-r}\{1 + 0(n^{-q})\}$, $r, q > 0$, then $s_n(B) = s_n(A)\{1 + O(n^{-\gamma})\}$, where $\gamma = \min\{q, ra(1 + ra)^{-1}\}$.*

REMARK 1. The estimate of the remainder is close to sharp: for the elliptic operators in $L^2(D)$ the remainder is of order given in Theorem 1.

THEOREM 2. *If $A > 0$, $\lambda_n(A) \sim cn^{-r}$ as $n \longrightarrow \infty$, $r > 0$, $|Qf| \leqslant c|A^af|$, $0 < a$, $N(I + Q) = \{0\}$, and $ra \geqslant 1$, then $B \in R_b(H)$.*

THEOREM 3. *If $l - m \geqslant d$ then $L + M \in R_b(H)$, $H = L^2(D)$. Furthermore* *if* $N(L+M) = \{0\}$, *then* $s_n(L+M) = s_n(L)\{1 + O(n^{-\gamma})\}$, *where*

$$\gamma = \min\{d^{-1}, (l-m)(l-m+d)^{-1}\}.$$

REMARK 2. If $d = 1$ then $m < l$ implies $l - m \geq 1$, and $L + M \in R_b(H)$.

4. Problems. (1) Let $Bf = \int_{-1}^{1} \exp\{i(x-y)^2\} f dy$ be an operator on $H = L^2([-1, 1])$. It is not known if $B \in R_b(H)$. (2) If $d > 1$ it seems to be an open problem if $L + M \in R(H)$ under the assumption of Theorem 2. Is the bracketing necessary? Some other problems can be found in [3, 5], where some questions of interest in applications are also discussed.

5. Comments. Minimax representation for $s_n(B)$ is the key point in the proof of Theorem 1. A proof of Theorem 2 can be based on a result from Appendix 11 in [3]. Theorem 3 can be derived from Theorem 2 and some known estimates for elliptic operators.

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