# GROTHENDIECK-RIEMANN-ROCH
# FOR COMPLEX MANIFOLDS

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For a coherent analytic sheaf $F$ on a complex manifold $X$ and a holomorphic map $f: X \longrightarrow Y$ to a complex manifold $Y$, with $f$ proper on the support of $F$, we prove a Grothendieck-Riemann-Roch formula as in [3] relating the Todd classes of $X$, $Y$ and the Chern character of $f$ and its direct images. This is the first example of a Riemann-Roch theorem for general mappings of complex manifolds which are not necessarily projective varieties. For the history of this problem see [2, 7, 8].

The theorem of Grauert shows that the direct image sheaves $R^i f_* F$ are coherent and we prove the formula

$$\sum_i (-1)^i \operatorname{ch}(R^i f_* F) \operatorname{Todd}(Y) = f_*(\operatorname{ch}(F) \operatorname{Todd}(X)).$$

The characteristic classes of $X$ and $Y$ are defined as in [1] and the Chern characters as in [9], so that all classes lie in Hodge cohomology. The theorem thus relates analytic, rather than topological, invariants of $F$ and its direct images. The same general methods should also lead to the corresponding statements for topological invariants in singular cohomology [2] and perhaps other cohomology theories. The de Rham Chern classes defined in [6] should be relevant in this context.

The proof is based throughout on the techniques of twisting cochains and local formulae introduced in [12] and developed in subsequent papers, and follows Grothendieck's approach to the extent that we factor $f$ as $\pi \circ \Gamma$ where $\Gamma: X \longrightarrow X \times Y$ is the graph of $f$ and $\pi: X \times Y \longrightarrow Y$ is the projection. We first prove the following two special cases.

THEOREM A. *Suppose $\iota: X \longrightarrow Z$ is a closed embedding with the property that there exists a holomorphic retraction $\rho: Z \longrightarrow X$ of maximal rank in a neighbourhood of $\iota(X)$. Then*

$$\operatorname{ch}(\iota_* F) = \iota_*(\operatorname{Todd}(N)^{-1} \operatorname{ch}(F))$$

*where $N$ is the normal bundle of $X$ in $Z$.*

THEOREM B. *Let $F$ be a coherent sheaf on $X \times Y$ with $\pi|\text{Supp }F$ proper.
Then*

$$\sum_i (-1)^i \text{ch}(R^i\pi_* F) = \int_X \text{Todd}(X)\text{ch}(F).$$

*The fibre integration makes sense because of the support condition, and gives
the Gysin map in this case.*

The general formula now follows by applying Theorem A with $Z = X \times Y$,
$\iota = \Gamma$ and retraction $\rho(x, y) = (x, f(x))$. Theorem B is then applied with $F$ re-
placed by $\Gamma_* F$.

The proof of Theorem A is straightforward and is based on the theory of
the Gysin map of [13] and a version of the trace map of [9]. We construct an
explicit twisted resolution of $\iota_* F$ which gives a local formula for $\text{ch}(\iota_* F)$, which
is then computed explicitly by reducing to the embedding of $X$ in $N$.

The heart of the matter lies in the proof of Theorem B. Locally with re-
spect to $Y$ this is essentially the same as the result of [10], the main difference
being that the finiteness and duality theorems and the Künneth formula are re-
placed by the corresponding relative versions. The finiteness result, proved as in
[4] for example, is the local existence of a complex of free $O_Y$-modules whose
cohomology gives the direct image sheaves. The duality theorem is given in [11]
and the Künneth formula also follows from the homological properties of topo-
logical vector spaces and completed tensor products.

The principal device used to make these results global on $Y$ is the 'shuffle
map' of Eilenberg-Mac Lane [5]. For Stein covers $U, V$ of $X$ and $Y$ and a
twisting cochain $(U \times V, F, b)$ for $F$ on $X \times Y$ the shuffle map $\phi$ applied to the
cochain $b$ on the product cover gives an 'infinite-dimensional twisting cochain'
$(U, F, \phi(b))$ on $Y$. The local complexes $\mathbf{F}_\alpha$ of this twisting cochain are com-
plexes of free nuclear Fréchet $O_Y$-modules in the sense of [4] and correspond
to the part of the cochain complex $C_b^\cdot(U \times V, F)$ lying over a fixed $V_\alpha$ in $V$.
This twisting cochain resolves the global complex $C^\cdot$ of $O_Y$-modules with $C^\cdot(V)$
$= C^\cdot(U \times V, F)$ whose cohomology gives the direct image sheaves. This also
supplies a natural Atiyah class for $C^\cdot$ which is the image of the $Y$-derivative part
of the Atiyah class of $F$ under the shuffle map.

The theory of the Gysin map applied as in [10] to the diagonal embedding
of $X \times Y$ into $X \times X \times Y$ then gives a formula

$$\text{ch}(C^\cdot) = \int_X \text{Todd}(X)\text{ch}(F)$$

where $\text{ch}(C^\cdot)$ is calculated using the powers of the Atiyah class and a trace map
obtained by using the Gysin map to represent the resulting cocycles as cocycles
of 'smoothing operators' between the local complexes $\mathbf{F}_\alpha^\cdot$. The proof of this

formula uses arguments similar to those applied in [10] to identify the restriction of the Lefschetz class to the diagonal.

Theorem B then follows from the relation

$$\text{ch}(C^{\cdot}) = \sum_i (-1)^i \text{ch}(R^i \pi_* F).$$

To prove this we develop a theory of twisting cochains and Chern character for complexes of sheaves, such as $C^{\cdot}$, which are 'perfect' in the sense of [7], i.e., are locally quasi-isomorphic to free finite complexes. For this definition of Chern character the formula above follows easily and so it remains to show that the two definitions agree. This can be reduced to the local Künneth formula and duality argument referred to above.

Other approaches to Theorem B are possible, but the one sketched is the simplest we have found.

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