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Nonlinear differential equations, by S. Fucik and A. Kufner, Studies in Applied Mechanics, vol. 2, Elsevier, New York, 1980, 359 pp., $83.00.

Recently three closely guarded secrets of modern mathematics and science have been revealed. They are

(i) the understanding of genuine nonlinear phenomena lies at the heart of many important problems in diverse areas of knowledge;

(ii) these nonlinear phenomena can often be adequately described by studying systems of nonlinear differential equations;

(iii) there are simple systematic mathematical ideas and techniques that are adequate to treat broad classes of these nonlinear systems. Moreover, when such ideas do not exist, they are being keenly pursued world-wide by many researchers, young and old.

Thus, each day seems to bring additional insights and significant mathematical results connecting the three facts mentioned above. These results are attained not only by professional mathematicians, but also by mathematically trained scientists and engineers whose work forces them to solve these problems.

All those who love mathematics have cause to rejoice since many modern mathematical areas developed until now for their own sake (e.g. homotopy groups of spheres and simple Lie groups, abelian functions, singularity theory, and the differential geometry of connections) are absolutely essential for the understanding of key nonlinear problems of science. These problems in turn spawn new fruitful directions of depth and subtlety for mathematics and science. Hidden links between diverse mathematical areas are being revealed. In short, we are witnessing the making of a new mathematical and scientific revolution.

The book under review explains some known (functional analysis) methods for certain classes of boundary value problems for certain nonlinear differential equations. The authors limit themselves to nonlinear elliptic equations and are thus able to treat ordinary and partial differential equations from a unified point of view. Before discussing the text, I believe it is prudent to discuss the background and contemporary developments of the subject.

I. Background. As mentioned previously, the present exciting state of affairs in nonlinear differential equations is due to the slow, discontinuous but combined efforts of many skillful investigators in many different lands. Its slow evolution in the twentieth century can be traced to many factors. First there is the hypnotic effect of the immensely successful linear (and linearizable) theories of Maxwell (electromagnetism and optics), Bohr-Schrodinger (quantum mechanics), Hodge-Kodaira (harmonic integrals), Dirac-Feymann (quantum electrodynamics), Schwartz (distributions), Hormander (linear partial differential equations), Oka-Cartan-Kohn (several complex variables) and Atiyah-Singer (index theory and elliptic topology) to mention only a few. As a rebellious student I once asked an emminent and eloquent spokesman of linear analysis if he were interested in nonlinear problems. "No, linear problems are hard enough," was the reply.

Secondly, books and survey articles were, and continue to be, written with a linear bias. With notable exceptions, most otherwise excellent books on functional analysis and partial differential equations not only stress exclusively linear problems but also neglect to mention the nonlinear aspects of their subject. The principle of superposition reigned supreme in physics, and where it didn't, that former great fountain of deep mathematical problems seemed to degenerate into jargonese. Only in the field of ordinary differential equations did books and articles contain systematic and deep nonlinear results. However, apart from celestial mechanics, these discussions were generally limited to problems involving phase plane techniques.

The cruel events of modern history also played a role in retarding the systematic study of nonlinear problems of analysis. For example, beginning in and continuing through the 1930's, the great Polish school of functional analysis led by Banach, Mazur, and Schauder carried out important research on nonlinear operators and nonlinear partial differential equations which was to culminate in a research monograph. What an influential sequel to Banach's *Théorie des Opérations Linéaires* this might have been! However, the book never appeared due to the destruction loosed by the Second World War. On a recent trip to Warsaw I inquired after the book but, sad to say, to no avail.

What then brought about this recent "nonlinear revolution", given all the forces working against it? The search for a greater, less restrictive vision of truth? Possibly. The search for a new unity and simplicity? Certainly. Rather than elaborate these basic abstract facts, I would also like to point to the recent solution of outstanding fundamental problems that have been not only spectacular and significant but also have opened promising developments for future research directions.

II. Contemporary developments. Samples of such interrelated developmnts connecting modern science and nonlinear differential equations include:

(A) *The determination of Einstein metrics on compact Kähler manifolds.* Nonlinear elliptic partial differential equations can be used quite effectively to deform given geometric structures on smooth compact manifolds, M^N, to simpler structures provided M^N is not too positively curved. One simply writes down a nonlinear partial differential equation (or system of equations) for the given deformation, taking care to introduce all the relevant geometry needed to simplify and restrict the problem. Then one attempts to find by analytic means a globally defined smooth solution for this equation, which in turn yields the desired deformation. Thus if $N = 2$, we can always conformally deform a given Riemannian metric, g, on M^2 to a Riemannian metric of constant Gaussian curvature, k, by solving a modified form of the nonlinear elliptic Liouville equation (if M^2 is simply connected, $k > 0$ and the partial differential equation problem is (I believe) still open). This leads to a new proof of the uniformization theorem for Riemann surfaces that is independent of certain covering space arguments. Progress in solving the higher-dimensional problem has not been easy due to the analytic difficulty of determining a priori estimates. However in recent years the combined efforts of E. Calabi, T. Aubin, and S. T. Yau have shown that for Kähler manifolds (M^{2n}, g) that are sufficiently negatively curved (so that their first Chern class is nonpositive) success can be achieved by solving nonlinear elliptic complex Monge-Ampère equations. In this work, the linear Laplace-Beltrami operator of the Liouville equation is simply replaced by the nonlinear complex Monge-Ampère operator. In fact if the first Chern class of M vanishes, the deformation preserves the cohomology class of g and yields an Einstein metric whose Ricci curvature vanishes identically. Similarly if the first Chern class of M is negative, the deformation yields an Einstein metric with scalar curvature -1. Yau went on to find new important applications of this work in algebraic geometry (the uniformization of certain algebraic surfaces and the result that every compact surface that is homotopic to CP^2 is biholomorphic to CP^2). Results on positive curvature cases pose a difficult problem since nonuniqueness and bifurcation phenomena must be overcome.

Science enters the picture because Einstein metrics are crucial for general relativity. Physicists generally attempt to find cleverly devised coordinate systems so that explicit solutions of Einstein equations can be written down (e.g., Schwartzschild and Kerr metrics). Modern science appears through quantum gravity and Euclidean gravity in which the nonlinear hyperbolic Einstein equations are analytically continued in the time variable to their elliptic analogues.

Thus we have the exciting convergence of two strands of knowledge. Without doubt this link promises an exciting and fruitful future.

For recent surveys and references for this research direction we refer readers to the articles of S. T. Yau and R. Penrose both appearing in Volume 1 of Proc. Internat. Congr. Math. (Helsinki, 1978), Acad. Sci. Fenn, Helsinki, 1980.

(B) *The complete integrability of nonlinear differential equations*. It is always important to find and solve special nonlinear differential equations as explicitly as possible using methods that are capable of generalization. One such approach is the classic method of Liouville (and its generalization due to V. Arnold) for Hamiltonian systems of finite dimension, $2N$, which involves finding N independent conserved integrals in involution. This leads to a global linearization (and explicit formulae for solutions) of the problem after appropriate canonical coordinate changes. Classic examples include geodesics on an ellipsoid (Jacobi), harmonic oscillators restricted to the sphere (Neumann), and motion of a top about a fixed point (Kowalevskaya). It is thus remarkable that recently this approach has not only yielded new finite-dimensional examples (e.g. Toda lattice), but has also been the fertile source of certain infinite-dimensional cases (i.e. two-dimensional completely integrable partial differential equations of Hamiltonian type). Notable partial differential equations are the Korteweg-de Vries equation, the sine-Gordon equation, and Boussinesq equations. All these possess soliton and multisoliton solutions (i.e. travelling wave solutions of permanent form moving with constant velocity). The complete integrability of such systems can be studied by Lie-theoretic techniques or inverse-scattering theory and in the case of periodic boundary conditions via the geometry of algebraic curves of finite and infinite genus. For full references to these researches, see the recent survey articles of McKean (Proc. ICM (Helsinki, 1978), vol. 2) Novikov (ibid, vol. 1) Lax (SIAM Review **18** (1976), 351–375) and Moser (Dynamical Systems, Birkhäuser, 1980).

Of course, the three nonlinear partial differential equations mentioned previously arise from different areas of nonlinear science. The remarkable prevalence and explicitness of soliton and multisoliton behaviour has excited almost all scientists interested in nonlinear phenomena and led to the discovery of soliton-like behaviour of other fields as well as to a study of the statistical mechanics and quantization of solitons for such systems.

Future directions for this work consist of (i) determining higher-dimensional partial differential equations of Hamiltonian type that are completely integrable and (ii) studying the stability of completely integrable systems under perturbations by new methods that do not break down when perturbed. There are some subtle points connected with (i) since generally such higher-dimensional physical problems possess at most a finite number of independent conserved integrals. What corresponds to solitons in these cases? For ideal fluids, steady vortex motions have been suggested by Zabusky and others.

Recently, for nonlinear elliptic problems, two other approaches to complete integrability might be mentioned. (i) Berger and Church showed that certain nonlinear Dirichlet problems defined over bounded domains Ω of arbitrary finite dimension are "globally conjugate" after canonical coordinate changes to an infinite-dimensional Whitney fold. This yields a new notion of complete integrability that provides stability results under perturbation and connects complete integrability with singularity theory.

(ii) Atiyah, Hitchin, Singer, Ward, Drinfeld, and Manin have shown that all absolute minima of the Euclidean Yang-Mills functional $S(A)$, otherwise known as instantons (see (C)), defined on S^4 can be obtained explicitly by interesting algebraic procedures, a remarkable achievement since the associated Euler-Lagrange equations of $S(A)$ appear as a formidable nonlinear elliptic system.

(C) *Nonlinear gauge theories*. How does one make the highly successful linear Maxwell equations of electromagnetism into an equally successful genuinely nonlinear theory? There are numerous interesting answers to this question, but the one I shall now describe appears to many mathematicians to be of particular depth, attractiveness, and challenge, viz., Euclidean quantum field theory with nonabelian gauge group G. Such theories can be understood by combining the Feynmann path integral approach to quantization, Schwinger's ideas concerning Euclidean field theory, and the notion of semiclassical approximation of the associated path intgegral. The result of all this from the point of view of nonlinear differential equations is the necessity of determining the smooth critical points of a nonquadratic action functional $S(A)$ defined on R^4 that has invariant properties with respect to a given compact Lie group G (called its gauge group). The associated Euler-Lagrange equations are nonlinear and can be chosen to be elliptic. The physical justification for the choice of "Euclidean" instead of the more obvious "Minkowskian" field theory takes considerable physical insight and is well described in recent Erice lecture notes of S. Adler, where the nonabelian nature of the gauge group G is strongly utilized. The actual gauge invariant form of $S(A)$ (without considering external sources) can be determined from standard functionals by a well-known "gauge principle". In the simplest case $S(A)$ can be interpreted differential geometrically as the L_2 norm of the curvature of a connection A associated with the group G. However, when we consider the boundary conditions for the differential forms A, we find important topological effects entering the problem (in the physics literature these effects are termed "nonperturbative" since they cannot be obtained by standard linearization arguments). To my knowledge these effects appeared first in the important work of Abrikosov of 1957 on vortices in Type II superconductivity where the term "flux quantization" is used. The simple case of $S(A)$ is usually referred to as pure Yang-Mills theory with $G = SU(2)$. The absolute minima of this functional known as "instantons" were discussed briefly in (B) and have been explicitly determined after extremely interesting preliminary work by Atiyah, Singer, Hitchin, Schwartz, Jackiw, and Polyakov, to mention only a few. Nonabsolute minima of $S(A)$, even in the simple cases, still form an enigma. (See recent work of Atiyah-Bott.) Such saddle points are often crucial in nonlinear problems.

However we cannot expect theoretical physics to conform to strict differential geometry. New boundary conditions, radiative corrections, external sources, alternative forms of the action functional $S(A)$ involving critical parameter dependence, will undoubtedly play a crucial role in future developments. The actual solution of fixed physical problems and prediction of new verifiable effects hopefully should involve joint efforts of mathematicians and physicists. A good example is the still unsolved "quark confinement" problem.

Recent surveys and references can be found in Atiyah (Proc. ICM (Helsinki, 1978), vol. 1) and Jaffe (ibid).

For lack of space I will not discuss the remarkable recent researches on (to mention only a few)

(D) Periodic solutions (local and global) of nonlinear conservative and nonconservative dynamical systems for both ordinary and partial differential equations.

(E) Global elliptic free boundary problems.

(F) Bifurcation phenomena.

(G) Nonlinear differential equations and geometric measure theory.

Suffice it to say that the results obtained so far have been far beyond expectations (resulting in the solution of many classic problems often by very beautiful and powerful new techniques).

References for many of the topics mentioned previously can be found in the Proceedings of the ICM (Helsinki, 1978), in particular the articles of Almgren, Rabinowitz, Clarke, Bona, and Raviart (all appearing in volume 2) and the books of Sattinger (*Group theoretic methods in bifurcation theory*, Lecture Notes in Math., vol. 762, Springer-Verlag, Berlin and New York, 1979), Nirenberg, *Nonlinear functional analysis* (Courant Institute Lecture Notes, 1974) and the reviewer, *Nonlinearity and functional analysis*, Academic Press, New York, 1981 (3rd reprinting with supplements).

III. The book. The book by Fucik and Kufner is intended for a broad class of readers, nonspecialist mathematicians and engineers, and thus requires only a modest mathematical background. (Elementary functional analysis including the Lebesgue integration would suffice.) The authors are not fussy about giving details of proof, systematic treatment of the material, or of the detailed applications to physical problems. The text introduces mathematical background facts as needed, but runs into the usual difficulties, that is, central elementary facts such as the implicit function theorem are introduced only toward the end of the book.

The book is well printed and makes pleasant reading. The text begins with nonlinear elliptic boundary value problem examples from engineering and discusses Sobolev spaces and weak solutions of nonlinear differential equations with numerous examples. Minimization methods utilizing the direct method of the calculus of variations based on Sobolev spaces and the Leray-Schauder degree are the basic methods of solution discussed. There is also a discussion of the regularity of weak solutions. The last two chapters are more novel. Chapter 6 treats noncoercive boundary value problems for semilinear second order elliptic partial differential equations (a case in which the usual methods break down). A rudimentary classification and a nonlinear Fredholm alternative is attempted. The text ends with a nice exposition of variational inequalities and their applications. All in all, this is a good introductory discussion of nonlinear elliptic problems with numerous examples included. For deeper mathematical ideas and directions for future developments such as those described previously, the reader will, however, have to look elsewhere.

To the authors' credit, incorrect statements are rare (exceptions include the discussion of weak solutions of semilinear nonlinear problems with exponential growth, page 146). Of course a book written at this level cannot cover all of the basic important topics. The reader should note that bifurcation theory, nonlinear eigenvalue problems, notions of Fredholm operators, periodic solutions of Hamiltonian systems, saddle points of nonquadratic functionals are among the important topics not discussed.

The mathematical community owes a debt of thanks to Fucik and Kufner, the two Czech authors of this book. They have produced a readable account of important contemporary topics in nonlinear analysis. These days, so much important research of our best people is dribbled out of them, piecemeal, in the form of imperfectly developed journal articles and conference proceedings. Let us hope that in the near future, other highly talented mathematicians of nonlinear science will be afforded the opportunity and leisure to share with us their finest conceptions in the form of systematically developed books, accessible to a wide mathematically educated audience.

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The symmetric eigenvalue problem, by Beresford N. Parlett, Prentice-Hall Series in Computational Mathematics, Prentice-Hall, Englewood Cliffs, N. J., 1980, xix + 348 pp., $ 25.00 cloth.

The thesis of this review may be summarized in three propositions. First, numerical analysis is a science with mathematical, empirical, and engineering components. Second, a conventional mathematical education does not equip one to deal with the last two components. Third, the book under review is a good place for a mature mathematician to get an appreciation of all three aspects of the subject.

At the outset I would like to correct a possible misapprehension. For most of this essay, I am going to focus on the nonmathematical aspects of numerical analysis. This does not mean that I wish to minimize the role of mathematics in numerical analysis; on the contrary, it is hard to overstate its importance. But the pure mathematician coming to the field for the first time will find much that is strange, and I hope this review will provide a brief guide to this extra-mathematical territory.

The mathematical component of numerical analysis scarcely needs arguing. The subject derives its analytic tools from many branches of mathematics. Its journals usually present results in the form of theorems, the coin by which mathematical productivity is currently measured. Nor are these theorems more trivial or less rigorously established than those of other branches of mathematics. Finally, most numerical analysis courses are listed in mathematics departments, perhaps jointly with a computer science department.

The empirical component of numerical analysis derives from the fact that numerical analysis is a branch of applied mathematics, and its results are therefore subject to outside verification. In general, an applied mathematician must look on a piece of experimental apparatus with a mixture of hope and trepidation, since it can confirm or deny his researches with unarguable