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ON THE LOCAL MONODROMY OF A VARIATION OF HODGE STRUCTURE

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Associated to a variation of polarized Hodge structure there is a period mapping $\psi: S \longrightarrow \Gamma\backslash D$, where S is the parameter space and $\Gamma\backslash D$ denotes the corresponding modular variety of polarized Hodge structures (the primary example to keep in mind is that arising from a family of smooth projective varieties parametrized by S) [3], [4]. The local study of the singularities of ψ ([5]) reduces to the case when $S = (\Delta^*)^l \times \Delta^m$, a product of punctured disks and disks.

Given a lifting $\tilde{\psi}: U^l \times \Delta^m \longrightarrow D$ (U = upper half-plane) of ψ to the universal covering of S there are monodromy transformations $\gamma_1, \ldots, \gamma_l \in \Gamma$ such that

$$\tilde{\psi}(z_1, \ldots, z_i + 1, \ldots, z_l; w_1, \ldots, w_m) = \gamma_i \tilde{\psi}(z_1, \ldots, z_i, \ldots, z_l; w_1, \ldots, w_m).$$

These γ_i's, which are quasi-unipotent automorphisms of the **C**-vector space H underlying the variation, provide important invariants of the singularities of ψ. In particular, in the single variable case ($l = 1, m = 0$) a central role is played by the monodromy weight filtration $W_* = W_*(N)$ of the nilpotent transformation $N = \log \gamma_u$, where γ_u is the unipotent part of the monodromy γ. We recall [5] that, if k is the weight of the Hodge structures, $N^{k+1} = 0$ and the filtration $(0) \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H$ is uniquely characterized by the conditions $NW_j \subseteq W_{j-2}$ and $N^j: W_{k+j}/W_{k+j-1} \longrightarrow W_{k-j}/W_{k-j-1}$ is an isomorphism.

The results announced here concern the monodromy weight filtrations arising in the several variables case. The main statements–Theorems 1 and 2–were conjectured by P. Deligne [2] (cf. Conjecture 1.9.6, as well as Theorem 1.9.2 for the special geometric case). For structures of weight two they are contained in [1].

THEOREM 1. *Let $\gamma_1, \ldots, \gamma_l$ be monodromy transformations of a period mapping $\psi: (\Delta^*)^l \times (\Delta)^m \longrightarrow \Gamma\backslash D$ and N_i the logarithm of the unipotent part*

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of γ_i. Then all the elements in the open cone of commuting nilpotent endomorphisms

$$\sigma = \left\{ \sum_{i=1}^{l} \lambda_i N_i;\ \lambda_i \in \mathbf{R},\ \lambda_i > 0 \right\}$$

define the same monodromy weight filtration (to be denoted by $W_*(\sigma)$).

According to Schmid's Nilpotent Orbit Theorem [5] there exists a limiting Hodge filtration F^* (depending on $w_1, \ldots, w_m$) such that the lifting $\widetilde{\psi}$ may be approximated for $\operatorname{Im} z_j >> 0$ by the orbit $\exp(\Sigma_j z_j N_j) \cdot F^*$. Moreover in the case of a single N and as a consequence of the SL_2-orbit theorem, $(W_*(N), F^*)$ defines a polarized mixed Hodge structure, i.e. the Hodge filtration F^* defines Hodge structures on the graded quotients $\operatorname{Gr}_j(N) = W_j(N)/W_{j-1}(N)$ which are suitably polarized (cf. Theorem 6.16 in [5] for the precise statement, as well as [6]). When this is combined with Theorem 1 one obtains that every point in the approximating orbit defines a mixed Hodge structure relative to the filtration $W_*(\sigma)$.

A further consequence of Theorem 1 is the existence of a Hodge filtration F_0^* such that $(W_*(\sigma), F_0^*)$ is a mixed Hodge structure *split over* $\mathbf{R}$ and the orbit $\exp(\Sigma_j z_j N_j) \cdot F_0^*$ lies in D for $\operatorname{Im} z_j > 0$. For a single monodromy transformation these "split" nilpotent orbits correspond to SL_2-orbits in the sense of [5]. Thus they could be expected to play a role in extending the SL_2-orbit theorem to the case of period mappings of several variables.

The second theorem relates the weight filtration $W_*(\sigma)$ to those associated to the faces of the cone σ. In order to make this statement precise let us consider two commuting nilpotent transformations N and N'. Since N' preserves $W_*(N)$, it induces nilpotent endomorphisms in each of the graded quotients $\operatorname{Gr}_j(N)$. Suppose that the monodromy weight filtrations defined by N' in the various $\operatorname{Gr}_j(N)$'s are all projections of a single filtration W_* of the total space with the property that $N'W_j \subseteq W_{j-2}$ (such a W_* need not exist for an arbitrary commuting pair N, N', but if it does, it is unique [2]). Then, following Deligne, we call such W_* the monodromy weight filtration *of the pair* $(N', W_*(N))$.

THEOREM 2. *Let N, N' be any two elements in the closure of σ not belonging to the same proper face of this cone. Then $W_*(\sigma)$ is the monodromy weight filtration of the pair $(N', W_*(N))$.*

We end with a bare bones sketch of the proof of Theorem 1. For elementary reasons there exists a Zariski-open subcone σ' of σ where the map $N \longrightarrow W_*(N)$ takes values in a fixed flag manifold. Then the existence of a Hodge filtration F^* such that $(W_*(N), F^*)$ is a polarized mixed Hodge structure for all $N \in \sigma$ is seen to imply the vanishing of the differential of that map. Hence the filtration $W_*(N)$ must be constant on each connected component of σ'.

For any such component σ_0' it is now possible to construct a new Hodge filtration F_0^* such that $(W_*(\sigma_0'), F_0^*)$ is a mixed Hodge structure split over $\mathbf{R}$ and such that $\exp zN \cdot F_0^* \in D$ for $N \in \sigma_0'$ and $\operatorname{Im} z > 0$. This, combined with Lemma 1 in [1], guarantees in turn that $(W_*(N), F_0^*)$ is a mixed Hodge structure even for an N in the closure of σ_0' in σ. The conclusion that $W_*(N) = W_*(\sigma_0')$ for any such N and, hence, that of Theorem 1, follows from the Proposition below which, together with the reduction to the $\mathbf{R}$-split case, also underlies the proof of Theorem 2.

PROPOSITION. *Let (W_*, F^*) be a mixed Hodge structure split over $\mathbf{R}$ and N a $(-1, -1)$-morphism of it. Then $(W_*(N), F^*)$ is a mixed Hodge structure if and only if $W_*(N) = W_*$.*

Detailed proofs will appear elsewhere. We wish to thank Pierre Deligne for his generous advice and encouragement during the preparation of this work.

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