15. Ju. G. Reshetnyak, *On the stability of conformal mappings in multi-dimensional spaces*, Sib. Mat. Zhurnal **8** (1967), 91–114.(Russian)

16. O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Preuss. Akad. Wiss. **22** (1940), 1–197.

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Optimization and approximation, by W. Krabs, translated by Philip M. Anselone and Ronald B. Guenther, John Wiley and Sons, Chichester-New York-Brisbane-Toronto, 1979, xii + 220 pp., $28.95.

0. Introduction. Although approximation theory is a discipline with very wide spectrum overlapping with areas ranging from classical function theory to operator theory, the very basic problems are those that deal with best approximation. More precisely, questions of existence, uniqueness, and characterization of the best approximants from a certain subspace or a certain submanifold, as well as the questions of determination and estimation of the orders of best approximation are of utmost importance. Very frequently, with the possible exception of the problem of uniqueness, the techniques and theory of optimization turn out to be quite useful. This book is devoted to the connection between best approximation and optimization; it contains numerous examples showing how approximation problems of different types arising in physical as well as technical applications can be treated in a unified way within the general framework of optimization theory. The main body of the book consists of three chapters: Chapter I treats the general linear theory of optimization including the standard existence and duality theorems, while Chapters II and III deal with the convex problems and the general nonlinear problems respectively. Each of these chapters begins with a series of examples followed by the development of the theoretical principles. The advantage of this presentation is that each chapter can be read almost independently. An appendix on the background of functional analysis is also included. We will divide our discussion into three parts, each of which deals with one chapter of the book.

I. Linear problems. The general topic treated throughout this book is that of continuous approximation problems. These lead to semi-infinite (completely infinite) optimization problems in which finitely (infinitely) many free parameters can occur along with infinitely many side conditions. Such infinite optimization problems arise when one seeks to approximate a continuous real-valued function f on a compact set S by linear combinations of continuous functions on S as well as possible in the sense of the maximum norm. As the author mentions, such a situation occurs when one seeks simplified representations of functions for the purpose of evaluation on a computer. This situation also occurs in the approximate solution of boundary and initial value problems for ordinary and partial differential equations.

There are many such examples scattered throughout the book. A typical problem, as given by Krabs, concerns mathematically treating the problem of air pollution. We include the example here since it is in the same vein as many of the other examples.

In a given two-dimensional control region S, a certain air quality is to be guaranteed. At the same time, the yearly average concentration of a pollutant (e.g. sulphur dioxide or carbon monoxide) is to be kept below a prescribed standard which is described by a real-valued function ϕ on S. The concentration observed in S is assumed to come from two kinds of sources: (a) sources which can be controlled and hence regulated, and (b) sources which cannot be controlled.

If, say, n controllable sources are present, then it is assumed that these contribute a yearly average $u_1, \ldots, u_n$ to the air pollution. We denote by u_0 the contribution from the uncontrollable sources. Then $u_0, u_1, \ldots, u_n$ are again real-valued functions on S whose actual determination as a rule is naturally a very difficult problem. From the requirement that the concentration ϕ is not to be exceeded, we are first led to the side conditions $\sum_{j=1}^{n} u_j(s) + u_0(s) \leqslant \phi(s)$ for all $s \in S$. If any of these conditions is not satisfied, then the contributions of the controllable sources are reduced and, in fact, the jth source is reduced by a factor x_j, $0 \leqslant x_j \leqslant 1$ for $j = 1, \ldots, n$, so that the side conditions

$$\sum_{j=1}^{n} (1 - x_j)u_j(s) + u_0(s) \leqslant \phi(s) \quad \text{for all } s \in S \tag{$*$}$$

are satisfied. Obviously these will be satisfied, if, for all $s \in S$, the condition $u_0(s) \leqslant \phi(s)$ is satisfied.

If any $x_j \neq 0$ must be chosen, then, in general, costs naturally arise (e.g. by changing production plans, introducing air purifiers, etc.). In the simplest case these costs are assumed to be proportional to x_j. Let c_j be the proportionality constant, so that with the choice of reduction factors $x_1, \ldots, x_n \in [0, 1]$ the total cost is

$$c(x_1, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j. \tag{$**$}$$

One now tries to choose the factors $x_1, \ldots, x_n$ subject to the side conditions $(*)$ so that the cost $c(x_1, \ldots, x_n)$ is as small as possible.

The author gives several more interesting examples, most of them related to boundary value problems, before getting to the theory. Among the most interesting is the relation drawn between certain boundary value problems of monotonic type and one-sided uniform approximation.

Krabs next introduces the "general linear optimization problem" which is essentially the linear programming problem. That is: let E and F be two partially ordered normed vector spaces. Let a continuous linear form C: $E \to R$ and a fixed element $b \in F$ be given. The general linear optimization problem (GLOP) is:

$$\text{minimize } C(x) \text{ subject to the side conditions } A(x) \geqslant b,\ x \geqslant \theta_E.$$

Along with the GLOP the corresponding dual problem is stated. The author then proceeds to give many of the highlights of the theory. This includes the weak duality theorem, the complementary slackness theorem, subconsistency and normality of the general linear optimization problem, and general existence theorems for the GLOP and the dual problem. Also Krabs gives limits as to how far one can extend solving the GLOP. Most notable is an example of Kretschmer [4] on the appearance of a duality gap which seems to be the first one in the literature about infinite linear optimization problems. The author is not so much worried about completeness of the theory as he is about showing immediate applications of the theorems. For example he gives an example of a boundary value problem for the potential equation. He then uses the weak duality theorem to get an optimal bound to the corresponding maximum problem and an abstract characterization theorem to verify that his "guess" of an optimal approximate solution is indeed optimal.

II. Convex problems. The linear problems and theory discussed in Chapter 1 are now generalized to the convex setting. To facilitate the presentation, the general linear approximation problem is discussed, followed by results on optimal error estimates. A typical example from optimal control is included, namely: Consider the differential equation $\ddot{x} + a\dot{x} + bx = u$ with $x(0)$ and $\dot{x}(0)$ prescribed, and the control function u satisfying $|u(t)| \leq c(t)$, where $c \in C[0, T]$. Given a trajectory k, the minimization problem is to study $\inf\{\|x_n - k\|_2: x_n$ is a solution of the initial value problem with $|u| \leq c\}$. This convex approximation problem arises from a forced vibration process.

The theory of convex functionals and convex mappings is appropriately introduced here, and this leads naturally to the existence and duality theorems in convex optimization. As we pointed out in our discussion of Chapter 1, Krabs is primarily concerned with infinite dimensional problems. Hence, the beautiful theory of finite dimensional convex optimization problems treated by books such as Karlin [2], Rockafeller [9], Stoer and Witzgall [10] is not included here. Ever since von Neumann proved his celebrated Saddle Point Theorem [11], many generalizations to infinite dimensional spaces were obtained. In particular, the generalized Kuhn-Tucker Theorem is discussed in this chapter.

Perhaps the most interesting section in this chapter is the one on application to approximation problems. Here, several characterization theorems are established, and the calculation of minimal deviation is derived. The final section deals with convex optimization problems in function spaces. Here again the problem is posed and optimality characterizations are given. As an example, the semi-infinite problem in the control of air pollution is taken up once more. This problem, as discussed in I, is to minimize the total cost function given by the functional (**). If S is assumed to be closed and bounded and the functions $u_1, \ldots, u_n, \phi$ are continuous on S, and $u_0(s) < \phi(s)$, $s \in S$, then the minimum solution exists, and in fact, the solution set has explicit forms. The last condition that $u_0(s) < \phi(s)$, $s \in S$, means that the pollution originating from an uncontrollable source lies in the whole region S under prescribed standard. The author uses this example to motivate a theorem on the optimal solution of a mixed linear convex problem.

III. Nonlinear problems. The last chapter of the book deals with nonlinear optimization problems. The standard nonlinear approximation problem, as stated by Krabs, is to minimize a convex functional over a nonempty subset of a normed vector space E. The best known example of this is rational approximation in which the general setting consists of two linear subspaces U and W of $C(M)$ of dimensions $(r + 1)$ and $(s + 1)$ $(r, s \geqslant 0)$ which are spanned by the functions $u_0, \ldots, u_r \in C(M)$ and $w_0, \ldots, w_s \in C(M)$. If one assumes that the convex cone

$$W^+ = \{w \in W\colon w(t) > 0 \text{ for all } t \in M\}$$

is not empty and defines

$$V = \{u/w\colon u \in U \text{ and } w \in W^+\},$$

the general rational approximation problem consists of approximating a function $x \in C(M)$ as well as possible in the sense of the maximum norm by quotients $u/w \in V$. Also mentioned are several examples of nonlinear boundary value problems which can be recast as nonlinear optimization problems having an infinite number of constraints.

The main theoretical contribution of this section lies in deriving necessary conditions for identifying minimal points as solutions of the unconstrained minimization problem. For the nonlinear approximation problem, this leads to the generalized Kolmogoroff condition of Meinardus and Schwedt [3], [7]. The necessary condition states that a convex functional f is minimized at x_0 over a subset X if f is minimized at x_0 with respect to variation in the tangent cone. Krabs then shows that under certain smoothness assumptions on the set X, sufficiency conditions for attaining a minimum can be derived. The final section of the chapter is concerned with nonlinear optimization with an infinite number of side conditions where a generalized Lagrange multiplier rule is derived.

In this chapter, as before, Krabs is more concerned with giving applications of the central part of the theory rather than delving into all of the theoretical ramifications.

CONCLUDING REMARKS. There are several books that deal with infinite optimization and its applications to problems in approximation theory, namely, Holmes [1], Laurent [5], Luenberger [6], and Pschenitschny [8]. While each of these books has its own desirable features, Krabs' book is quite different from them. It is, in fact, a good supplement to the existing literature, since it provides a good source of practical examples and since each of the three chapters can be read almost independently. In short, it is quite suitable as a reference book both for researchers and for beginners who wish to understand the relationship between optimization and approximation.

REFERENCES

1. R. B. Holmes, *A course on optimization and best approximation*, Lecture Notes in Math., vol. 257, Springer-Verlag, Berlin and New York, 1972.

2. S. Karlin, *Mathematical methods and theory in games*, Programming and Economics, 2 vols., Reading, Mass., 1959.

3. A. N. Kolmogorov, *A remark on the polynomials of P. L. Čebysev deviating the least from a given function*, Uspehi Mat. Nauk (N.S.) 3, no. 1 (23) (1948), 216–221. (Russian)

4. K. S. Kretschmer, *Programs in paired spaces*, Canad. J. Math. 13 (1961), 221–238.

5. P.-J. Laurent, *Approximation et optimisation*, Hermann, Paris, 1972.

6. D. G. Luenberger, *Optimization by vector space methods*, Wiley, New York, 1969.

7. G. Meinardus and D. Schwedt, *Nicht-lineare Approximationen*, Arch. Rational Mech. Anal. **17** (1964), 297–326.

8. B. N. Pschenitschny, *Notwendige Optimalitätsbegingungen*, München-Wien, 1972.

9. R. T. Rockafellar, *Duality and stability in extremum problems involving convex functions*, Pacific J. Math. **21** (1967), 167–187.

10. J. Stoer and Chr. Witzgall, *Convexity and optimization in finite dimensions*. I, Die Grundlehren der Mathematischen Wissenschaften, Band 163, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

11. J. von Neumann, *Zur Theorie der Gessllschaftsspiele*, Math. Ann. **100** (1928), 295–320.

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Lectures on pseudo-differential operators: *Regularity Theorems and applications to non-elliptic problems*, by Alexander Nagel and E. M. Stein, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1979, 159 pp., $6.75.

Pseudodifferential operators may be considered from the ontological, the teleological, or the archeological standpoint: what are they, what do they do, where do they come from? Quick answers are that they are linear operators expressed via the Fourier transform as (formal) integral operators, that they are used extensively in the study of partial differential equations, and that the direct line of descent is through singular integrals. We shall consider each point in more detail and detect as well a thread of Hegelian dialectic.

If $P = \Sigma a_\alpha(x)D^\alpha$ is a differential operator in R^n and e_ξ denotes the exponential $e_\xi(x) = \exp(ix \cdot \xi)$, then $Pe_\xi(x) = p(x, \xi)e_\xi(x)$, where $p(x, \xi) = \Sigma a_\alpha(x)\xi^\alpha$. Expressing a test function as a sum of exponentials by the Fourier inversion formula, one obtains

$$Pu(x) = \int e^{ix \cdot \xi} p(x, \xi)(\xi)\, d'\xi = \int\int e^{ix \cdot \xi} p(x, \xi) u(y)\, dy\, d'\xi, \tag{1}$$

where $d'\xi = (2\pi)^{-n}\, d\xi$. Thus the differential operator P is expressed formally as an integral operator defined by a conditionally convergent "oscillatory" integral by means of the "symbol" p. Here $p = p_r + p_{r-1} + \cdots$, where p_j is homogeneous of degree j in ξ. Thus at least locally in x one has estimates for the derivatives

$$\left|D_x^\beta D_\xi^\alpha\left(p - \sum_{k>j} p_k\right)\right| < C_{\alpha\beta j}|\xi|^{j-|\alpha|} \quad \text{if } |\xi| > 1. \tag{2}$$

These estimates may be used in conjunction with integration by parts in (1) to convert (1) into a convergent integral and verify directly that P maps $C_c^\infty(R^n)$ to $C^\infty(R^n)$. Now if Q is a second differential operator, with symbol $q = q_s + q_{s-1} + \cdots$, then the composition QP has symbol which can be calculated by Leibniz' rule:

$$q \circ p = \sum (\alpha!)^{-1} i^{-|\alpha|} D_\xi^\alpha q D_x^\alpha p. \tag{3}$$

In particular the highest order part is just the product $p_r q_s$. Note in passing that the identity operator has symbol 1.