

AFFINE MANIFOLDS AND SOLVABLE GROUPS

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Let M be a compact affine manifold. Thus M has a distinguished atlas whose coordinate changes are locally in $\text{Aff}(E)$, the group of affine automorphisms of Euclidean n -space E . Assume M is connected and without boundary.

The universal covering \tilde{M} of M has an affine immersion $D: \tilde{M} \rightarrow E$ which is unique up to composition with elements of $\text{Aff}(E)$. Corresponding to D there is a homomorphism $\alpha: \pi \rightarrow \text{Aff}(E)$, where π is the group of deck transformations of \tilde{M} , such that D is equivariant for α . Set $\alpha(\pi) = \Gamma$. Let $L: \text{Aff}(E) \rightarrow GL(E)$ be the natural map.

THEOREM 1. *If Γ is nilpotent the following are equivalent:*

- (a) M is complete, i.e. $D: \tilde{M} \rightarrow E$ is bijective;
- (b) D is surjective;
- (c) no proper affine subspace of E is invariant under Γ ;
- (d) $L(\Gamma)$ is unipotent;
- (e) M has parallel volume, i.e. $L(\Gamma) \subset SL(E)$;
- (f) M is affinely isomorphic to $\Gamma \backslash G$ where G is a connected Lie group with a left-invariant affine structure and $\Gamma \subset G$ is a discrete subgroup;
- (g) each de Rham cohomology class of M is represented by a differential form whose components in affine charts are polynomials.

For abelian Γ the equivalence of (a), (d), and (e) is due to J. Smillie. We conjecture that (a), (b), (e), and (g) are equivalent even without nilpotence (if M is orientable). In general (a) \Rightarrow (c) and (e) \Rightarrow (c); but (c) $\not\Rightarrow$ (a) even for Γ solvable and M three-dimensional.

THEOREM 2. *The following are equivalent:*

- (i) M is finitely covered by a complete affine nilmanifold M_1 (i.e. conditions (a) through (g) of Theorem 1 hold for M_1);
- (ii) all eigenvalues of elements of $L(\Gamma)$ have norm 1;
- (iii) M has a Riemannian metric whose coefficients in affine charts are polynomials.

L. Auslander has conjectured that if M is complete then $\pi = \Gamma = \pi_1(M)$ is virtually solvable (i.e. contains a solvable subgroup of finite index); see [M] for discussion. This conjecture is true in dimension three (see [FG]).

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THEOREM 3. *If π is virtually solvable and M is complete then (e), (f), (g) of Theorem 1 hold. If $\alpha: \pi \rightarrow \Gamma$ factors through a virtually polycyclic group of rank $\leq \dim M$ and M has parallel volume, then M is complete. In particular if M is finitely covered by a manifold homeomorphic to a solvmanifold then parallel volume is equivalent to completeness.*

We briefly indicate the proof of Theorem 1.

(a) \Rightarrow (c). This holds for any compact complete M . If $F \subset E$ is a Γ -invariant affine subspace then both E/Γ and F/Γ are Eilenberg-Mac Lane spaces of type $K(\pi, 1)$. Since they are compact manifolds their dimensions are equal; thus $F = E$.

(e) \Rightarrow (c). This holds for all compact M . The linear holonomy $\rho = L \circ \alpha: \pi \rightarrow GL(E)$ determines a π -module E_ρ . Let $u: \pi \rightarrow E$ send $g \in \pi$ into the translational part of $\alpha(g)$. Then u is a crossed homomorphism whose cohomology class $c_M \in H^1(\pi; E_\rho)$ depends only on M . The n th exterior power $\Lambda^n c_M \in H^n(\pi; \mathbf{R})$ vanishes if and only if M does not have parallel volume. On the other hand c_M can be expressed in de Rham cohomology of M with coefficients in E twisted by ρ . Suppose $F \subset E$ is Γ -invariant. We may assume F is a linear ρ -invariant subspace. Then c_M comes from $H^1(\pi; F_\rho)$. If $\dim F < n = \dim E$ then $\Lambda^n c_M$ comes from $H^1(\pi; \Lambda^n F) = 0$.

From now on assume Γ is nilpotent.

(c) \Rightarrow (d). Let $E_U \subset E$ be the maximal unipotent submodule. Then $H^0(\pi; E/E_U) = 0$, and nilpotent implies $H^1(\pi; E/E_U) = 0$ (Hirsch [H]). This means some coset of E_U is Γ -invariant.

(b) \Rightarrow (d). Suppose $E_U \neq E$. Some coset of E_U is Γ -invariant; we may assume E_U is Γ -invariant. There is a unique $L(\Gamma)$ -invariant splitting $E = E_U \oplus F$. Let $M_1 = p(D^{-1}E_U)$ where $p: \tilde{M} \rightarrow M$ is the projection. Then M_1 is a compact affine manifold with unipotent holonomy, hence complete. Let Y be the vector field on \tilde{M} which is D -related to the vector field $(x, y) \mapsto (0, y)$ on $E_U \oplus F$. Then Y covers a vector field on M , so Y is completely integrable. Every component of $p^{-1}M_1$ is a repeller for Y . One uses these facts to prove that M is complete; but this implies (c), and hence (d).

(d) \Rightarrow (a). When $L(\Gamma)$ is unipotent there is a flag $E = E_n \supset \cdots \supset E_0 = \{0\}$ of $L(\Gamma)$ -invariant linear subspaces with $L(\Gamma)$ acting trivially on each E_i/E_{i-1} . There are nested foliations Z_n, \dots, Z_0 on M covered by foliations \tilde{Z}_i on \tilde{M} such that D relates \tilde{Z}_i to the linear foliation E_i of E whose leaves are cosets of E_i . For each i there is a closed 1-form $\tilde{\omega}_i$ on \tilde{Z}_i which vanishes on \tilde{Z}_{i-1} , related by D to a constant 1-form on E vanishing on E_{i-1} . There are completely integrable vector fields X_i in \tilde{Z}_i with $\langle X_i, \tilde{\omega}_i \rangle = 1$. Given any $p \in \tilde{M}$, $x \in E$ one shows that the trajectory of X_n through p meets a point p_1 such that $D(p_1)$ is the leaf of E_{n-1} through x . The trajectory of X_{n-1} through p_1 stays in a leaf of \tilde{Z}_{n-1} and eventually meets a p_2 such that $D(p_2)$ is the leaf of E_{n-2} through x , etc. In this

way one proves that $D(\tilde{M})$ contains a path from $D(p)$ to x . Hence D is surjective. Injectivity is proved similarly.

(e) \Rightarrow (d). If $E_U \neq E$ let $F \subset E$ be a complementary submodule to E_U .

One shows that some element of $L(\Gamma)$ expands F , contradicting parallel volume.

(a) \Rightarrow (b) and (d) \Rightarrow (e) are obvious.

(a) \Rightarrow (f). G is the algebraic hull of Γ in $\text{Aff}(E)$.

(f) \Rightarrow (g). By Nomizu's theorem [N] the cohomology of M is represented by invariant forms on G ; these turn out to be polynomial.

(g) \Rightarrow (e). If $L(\Gamma)$ is not unipotent then one proves there is no polynomial volume form.

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