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BLOCKS WITH CYCLIC DEFECT GROUPS IN $GL(n, q)$

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Let G be a finite group and B an r-block of G with cyclic defect group R. The decomposition of the ordinary characters in B into modular characters is described by the Brauer tree T of B. The problem of determining the Brauer trees for finite groups of Chevalley type was proposed by Feit at the 1979 AMS Summer Institute. Our result is a necessary step in this problem: If $G = GL(n, q)$ and r is an odd prime not dividing q, then T is an open polygon with its exceptional vertex at one end. The proof also shows an interesting fit of the modular theory for such primes r with the underlying algebraic group, the Deligne-Lusztig theory, and Young diagrams.

Because R is a cyclic defect group, R has the form

$$R = \begin{pmatrix} I_l & 0 \\ 0 & R_1 \end{pmatrix}, \tag{1}$$

where the elementary divisors of a generator of R_1 are, say, m copies of an irreducible polynomial of degree d over F_q. By (1) the structure of $C = C_G(R)$ is

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & C_1 \end{pmatrix}, \tag{2}$$

where $C_0 \simeq GL(l, q)$ and $C_1 \simeq GL(m, q^d)$. The normalizer $N = N_G(R)$ is then obtained by adjoining to C an element t of the form

$$t = \begin{pmatrix} I_l & 0 \\ 0 & t_1 \end{pmatrix},$$

where t_1 induces a field automorphism of order d on C_1.

By Brauer's First Main Theorem B corresponds to a block B_C of C with defect group R, where B_C is determined up to conjugacy in N. Let E be the stabilizer of B_C in N, so $e = |E: C|$ is then the inertial index of B. Let Λ be a set of representatives for the orbits of E on the set of nontrivial irreducible characters of R. In the Brauer-Dade theory [1] the exceptional characters χ_λ in B are labeled by λ in Λ, the nonexceptional characters χ_i in B by $i = 1, 2, \ldots, e$, and the $e + 1$ vertices of T by $\chi_1, \chi_2, \ldots, \chi_e$, exc.

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The block B_C decomposes by (2) as $B_C = B_{C_0} \times B_{C_1}$, where B_{C_0} is a block of C_0 of defect 0 and B_{C_1} is a block of C_1 with defect group R_1. Since $R_1 \leqslant Z(C_1)$, a theorem of Reynolds [6] implies that the characters θ_λ in B_{C_1} can be labeled by the irreducible characters λ of R so that θ_1 is the unique r-rational character in B_{C_1} and so that $\theta_\lambda(xy) = \lambda(x)\theta_1(y)$ for any x in R_1 and any r'-element y of C_1.

Let $R_C^R(\theta)$ be the virtual character of G associated to the irreducible character θ of C by the Deligne-Lusztig theory. Thus $R_C^G(\theta)$ is an element of the Grothendieck ring of representations of G over $\overline{Q}_l$. (We have modified the notation of [2], [4]. We should strictly write $R_{\overline{C}}^{\overline{G}}(\theta)$, where $\overline{G}$ is the algebraic group $GL(n, \overline{F}_q)$, F is a Frobenius endomorphism of $\overline{G}$ with $G = \overline{G}^F$, $\overline{C}$ is a regular subgroup of $\overline{G}$, and $C = \overline{C}^F$.)

PROPOSITION 1. *There is a labeling of the* χ_λ, $\lambda \in \Lambda$, *and signs* ϵ_0, $\epsilon_1, \ldots, \epsilon_e$ *such that*

$$R_C^G(\theta_0\theta_\lambda) = \epsilon_0\chi_\lambda \quad \textit{for } \lambda \in \Lambda,$$

$$R_C^G(\theta_0\theta_1) = \epsilon_1\chi_1 + \epsilon_2\chi_2 + \cdots + \epsilon_e\chi_e.$$

The signs ϵ_i *and the generalized decomposition numbers of* B *corresponding to a generator* x *of* R *are related by*

$$\epsilon_i = d^x(\chi_i, \theta_0\theta_1) \quad \textit{for } 1 \leqslant i \leqslant e,$$

$$\epsilon_0 \sum_{g \in E/C} \lambda^g = d^x(\chi_\lambda, \theta_0\theta_1) \quad \textit{for } \lambda \in \Lambda.$$

A classification of the irreducible characters of G has been given by Green [3], and can be restated in the language of [5] as follows: Each irreducible character θ of G corresponds to an ordered pair (s, ξ), where s is a semisimple element of G and ξ is a unipotent irreducible character of $C_G(s)$. The correspondence is given by $\theta = \pm R_{C_G(s)}^G(\xi\eta)$, where η is the linear character of $C_G(s)$ dual to s. (The dual η of s is defined by fixing an isomorphism of $\overline{F}_q^*$ into $\overline{Q}_l$. Then $\mathrm{Hom}(\overline{L}^F/[\overline{L}, \overline{L}]^F, \overline{Q}_l^*) \simeq (Z(\overline{L}))^F$ for any regular subgroup $\overline{L}$ of $\overline{G}$. In particular, if s is a semisimple element of G and $\overline{L} = C_{\overline{G}}(s)$, then s is in $(Z(\overline{L}))^F$ and thus determines a linear character η of $C_G(s)$.)

Let (s_i, ξ_i) be a pair corresponding to the irreducible character θ_i of C_i for $i = 0, 1$. Thus

$$\theta_i = \pm R_{L_i}^{C_i}(\xi_i\eta_i),$$

where $L_i = C_{C_i}(s_i)$ and η_i is the linear character of L_i dual to s_i. In the case $i = 0$, ξ_0 has defect 0; in the case $i = 1$, L_1 is a torus of order $q^{dm} - 1$ in C_1 and ξ_1 is the 1-character of L_1. Let $s = s_0s_1$ and $K = C_G(s)$. It then follows that $C_K(R) = C_C(s) = L_0 \times L_1$, $N_K(R) = \langle C_K(R), t^f\rangle$, and $|N_K(R)/C_K(R)| = e$, where $d = ef$. In particular, if B_{L_0} is the block of L_0 containing ξ_0 and B_{L_1} is the principal block of L_1, then $(B_{L_0} \times B_{L_1})^K$ is a block B_K of K with defect group R and inertial index e. Let ψ_λ, $\lambda \in \Lambda$, be the exceptional characters in B_K, and $\psi_1, \psi_2, \ldots, \psi_e$ the nonexceptional characters in B_K. The $\psi_1, \psi_2, \ldots, \psi_e$ are then unipotent. Let η be the linear character of K dual to s.

PROPOSITION 2. *There is a labeling of the* $\psi_1, \psi_2, \ldots, \psi_e$ *and signs* $\delta_0, \delta_1, \ldots, \delta_e$ *such that*

$$R_K^G(\delta_0 \psi_\lambda \eta) = \epsilon_0 \chi_\lambda, \qquad \lambda \in \Lambda,$$

$$R_K^G(\delta_i \psi_i \eta) = \epsilon_i \chi_i, \qquad 1 \leqslant i \leqslant e.$$

In particular, R_K^G *induces a* 1-1 *correspondence between the nonexceptional characters of* B *and the unipotent characters in* B_K.

The group K is a direct product $K = K_1 \times \cdots \times K_t$, where K_i, say, is isomorphic to $GL(n_i, q^{d_i})$. Let $B_K = B_{K_1} \times \cdots \times B_{K_t}$ be the corresponding decomposition of B_K. We may label these so that B_{K_i} has defect 0 for $i > 1$. Then B_{K_1} has defect group R. The unipotent characters of K_1 are in natural 1-1 correspondence with the irreducible characters of the Weyl group W_1 of K_1, and since W_1 is the symmetric group on n_1 symbols, the unipotent characters of K_1 are thus naturally labeled by partitions of n_1. Since the characters in B_K are products of a fixed character of $K_2 \times \cdots \times K_t$ by the characters in B_{K_1}, we may also label the nonexceptional characters in B_K by the partitions $\mu_1, \mu_2, \ldots, \mu_e$ of n_1 labeling the unipotent characters in B_{K_1}. Thus we write ψ_{μ_i} for ψ_i.

PROPOSITION 3. *Each partition* μ_i *has a unique e-hook* ν_i and the hooks $\nu_1, \nu_2, \ldots, \nu_e$ are pairwise distinct. The e-core μ_0 of μ_1, obtained by deleting ν_i from μ_i, is the same for all i; moreover, μ_0 has no e-hooks. The μ_i account for all partitions μ of n_1 with a unique e-hook and e-core μ_0.

Let the partitions μ_i be arranged so that the e-hook ν_i of μ_i has leg length $i - 1$. By our labeling the nonexceptional characters in B are of the form $\chi_i = R_K^G(\delta_i \epsilon_i \psi_{\mu_i} \eta)$.

PROPOSITION 4. *If the* χ_i *are labeled as above, then the signs* δ_i *are given by* $\delta_i = (-1)^{i-1}$. *Moreover,* R_K^G *induces a graph isomorphism of the Brauer trees of* B_K *and* B. *The tree of* B *is the open polygon.*

$$\begin{array}{ccccccccc} \circ & \text{———} & \circ & \text{———} & \circ & \cdots & \circ & \text{———} & \circ \\ \chi_1 & & \chi_2 & & \chi_3 & & \chi_e & & \text{exc} \end{array} .$$

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