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A POINCARÉ-HOPF TYPE THEOREM FOR THE DE RHAM INVARIANT

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The Poincaré-Hopf theorem relates the Euler-characteristic of a manifold to the local behavior of a generic vector-field on the manifold in a neighborhood of its zeroes. As a corollary of this, by taking the gradient, one can calculate the Euler-characteristic of a manifold from a local knowledge of a generic map to R^1 around its singular points. We prove an analogue of this theorem for calculation of the de Rham invariant of $4k + 1$ dimensional orientable manifolds from a map to R^2.

For $4k + 1$ dimensional orientable manifolds we have the de Rham invariant $d(m)$. This invariant is

(a) the rank of the 2-torsion in $H_{2k}(M)$,

(b) $\hat{\chi}_Q(M) - \hat{\chi}_2(M)$ mod 2 where $\hat{\chi}_F(M)$ is the semicharacteristic of M with coefficients in F,

(c) $d(M) = [w_2 w_{4k-1}(M), [M]] = [v_{2k} sq^1 v_{2k}(M), [M]]$,

where $w_i(M)$ is the ith Stiefel-Whitney class and v_i is the ith *Wu* class of M.

For the equivalence of these definitions see [L-M-P]. The de Rham invariant is important in the theory of surgery; see [M] or [M-S].

Definition of the local invariant. Let M^m, N^n be C^∞ manifolds. Let $C^\infty(M, N)$ be the space of C^∞ maps from M to N topologized with the C^∞ topology. Within $C^\infty(M, N)$ we have a dense (in fact residual) subset $G(M, N)$ of maps which are generic in the sense of Thom-Boardman [B] and satisfy the normal crossing condition [G-G]. This second condition is essentially that f is in general position as a map of its singularity submanifolds to N.

Let $f \in G(M, R^2)$; then df is of rank 2 except on a collection of disjoint closed curves in M, the singular set of f, $S_1(f)$. At points of $S_1(f)$, df is of rank 1. Restricted to $S_1(F)$ f is an immersion except at a finite set of points, $S_{1,1}(f)$, the cusp points of f. $S_1(f) - S_{1,1}(f) = S_{1,0}(f)$ is the set of fold points of f. Suppose $x \in S_{1,0}(f)$ then we can choose coordinates $x_1, \ldots, x_n$ around x and coordinates y_1, y_2 around $f(x)$ so that

$$f(x_1, \ldots, x_n) = (x_1, x_2^2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).$$

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Similarly if x is a cusp point we can choose coordinates so that

$$f(x_1, \ldots, x_n) = (x_1, x_2^3 + x_1x_2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).$$

Now we quote a result from [L].

THEOREM. *Let M^m, $m > 2$ be of even Euler characteristic; then given $f \in G(M, R^2)$, f is homotopic to an f_1 in $G(M, R^2)$ with no cusp points.*

In the case that $f \in G(M, R^2)$ has no cusps $f | S_1(f)$ is an immersion. The normal crossing condition guarantees that $f(S_1(f))$ crosses itself in a finite number of double points with no triple points. Let

$$V(f) = \{y \in R^2 | f^{-1}(y) \cap S_1(f) = 2 \text{ points}\}.$$

Let $N(S_1(f), M)$ be the normal bundle to $S_1(f)$ in M and let G be the bundle over $S_1(f)$ defined by the following exact sequence:

$$T(M)|S_1(f) \xrightarrow{df} f^*T(R^2) \longrightarrow G \longrightarrow 0$$

where if M is a manifold $T(M)$ denotes its tangent bundle. Use of the (second) intrinsic derivative [L], [B] allows definition of a symmetric bilinear form

$$B\colon\ N(S_1(f), M) \otimes N(S_1(f), M) \longrightarrow G$$

on $N(S_1(f), M)$ with values in G. B is nondegenerate on $S_{1,0}(f)$ and has one-dimensional kernel on $S_{1,1,0}$. Given $x \in S_{1,0}(f)$ and a choice of an orientation of G_x, B_x is a nondegenerate bilinear form on $N(S_1(f), M)_x$. Define the absolute index $a(x)$ at x by $a(x) = \min(\text{index}(B_x), m - 1 - \text{index}(B_x))$. Note that $a(x)$ is independent of the choice of orientation of G_x.

Explicitly suppose C_j is a component of $S_1(f)$ with no cusps, then $f | C_j$ is an immersion so $N(f(C_j), R^2)$ the normal bundle to $F(C_j)$ in R^2 is well defined, isomorphic to G, and trivializable. Let $T\colon\ N(f(C_j), R^2) \longrightarrow R^1$ be a trivialization. Let $D(C_j, M)$ be a choice of normal disc bundle to C_j in M. By a slight abuse of notation we consider $f\colon\ D(C_j, M) \longrightarrow N(f(C_j), R^2)$; let $g_x\colon\ D_x(C_j, M) \longrightarrow R^1$ denote the function

$$g_x = T \circ f\colon\ D_x(C_j, M) \longrightarrow R^1.$$

Then $\{g_x | x \in C_j\}$ is a differentiable family of functions on the fibers of $D(C_j, M)$, each g_x has a Morse singularity at x and the form B_x is given by $d^2 g_x$.

Let M^m be orientable with $n = 2k + 1$; then M has zero Euler-characteristic so use the theorem of Levine to choose $f \in G(M, R^2)$ with no cusps. Let C_j be a component of $S_1(f)$; as M is orientable $N(C_j, M)$ is trivializable. Furthermore the choice of trivialization T above makes B a nondegenerate symmetric bilinear form on $N(C_j, M)$. Thus the structure group of $N(C_j, M)$ is given a reduction to $O^+(p, m - p - 1)$, the orientation preserving components of

$O(p, m-p-1)$. For $p \neq 0, m-1$, $O^+(p, m-p-1)$ has two components. Define $i(C_j) \in Z/2Z$, the index of C_j, by $i(C_j) = 0$ if and only if $N(C_j, M)$ is the trivial $O^+(p, m-p-1)$ bundle and $i(C_j) = 1$ if and only if $N(C_j, M)$ is the nontrivial $O^+(p, m-p-1)$ bundle. Note that $i(C_j)$ is independent of all choices of trivialization and orientations.

Define $\tau(f) \in Z/2Z$ by

$$\tau(f) = \sum i(C_j), \qquad C_j \text{ a component of } S_1(f).$$

Define $t(f) = |V(f)| \bmod 2$.

Statement of results.

PROPOSITION A. *$r(f) = t(f) + \tau(f)$ is independent of the choice of $f \in G(M, R^2)$ without cusps, so one can define $r(M) = r(f)$, $f \in G(M, R^2)$ without cusps.*

COMMENT. One can, with a little more effort, still define $r(f)$ for $f \in G(M, R^2)$ even when $f \in G(M, R^2)$ has cusps. However in this case $r(f)$ is no longer independent of f.

PROPOSITION B. *r is a homomorphism from oriented cobordism to $Z/2Z$; that is*

(a) *If $[M] = [N]$ in Ω_{2k+1} then $r(M) = r(N)$,*

(b) *$r(M_1 \cup M_2) = r(M_1) + r(M_2)$.*

Let $\chi(M)$ be the Euler characteristic of M reduced mod 2.

(c) *$r(M^{2k+1} \times N^{2p}) = r(M^{2k+1}) \cdot \chi(N^{2p})$.*

PROPOSITION C.

(a) *Let M^{4k+1} be an orientable manifold then $r(M) = d(M)$.*

(b) *Let M^{4k+3} be an orientable manifold then $r(M) = 0$.*

Thus Proposition C gives a way of determining the de Rham invariant, which is intersection theoretic in character, from the local behavior of a map M to R^2 around its singular set. It is illuminating to consider such an f as a pair of Morse functions in general position with respect to each other.

Sketch of proofs. Proposition A is proved by a careful analysis of a homotopy F from f_0 to f_1 where f_0 and f_1 are different choices of f on M^{2k+1}. We can take $F \in G(M \times I, R^2 \times I)$. First we reduce to the case that F has no dovetail singularities. In this case $S_1(F)$ is an embedded surface in $M \times I$ intersecting $M \times \{0\}$ and $M \times \{1\}$ normally in $S_1(f_0)$ and $S_1(f_1)$. On the interior of this surface we have circles of cusp points separating the surface into regions of constant absolute index. Let R_p be the union of the regions of absolute index p. Let $i(R_p) = \Sigma i(C)$ C a component of $\partial(R_p)$, then analysis of the cusp singularity yields the equation

$$\sum_{i=0}^{k} i(R_p) = \tau(f_0) + \tau(f_1).$$

For $p \neq k$ it is straightforward to prove $i(R_p) = 0$. As in the case that F has no dovetails we have

$$t(f_0) = t(f_1) \bmod 2.$$

Proposition A reduces to showing $i(R_k) = 0$. This is easy for k odd, but subtler for k even. For k even we prove

LEMMA. *Let P be a component of R_k; then P is a closed surface P^1 minus a collection of discs and P^1 is of even Euler characteristic. From this fact it follows from $i(R_k) = 0$.*

Proposition B is proved by first observing that $r(M)$ remains invariant if M is cut open and repasted along a codimension 1 submanifold of the form $S^1 \times F$ by a pasting ϕ: $S^1 \times F \longrightarrow S^1 \times F$ with $\phi(x \times F) = x \times F$. Given this observation the results of [A] allow immediate demonstration of bordism invariance. For the relation of cutting and pasting and cobordism see [K-K-N-0].

Proposition C follows from explicit construction of the examples (using, for instance, (c) of Proposition B) in each dimension $4k + 1$ on which r and d agree and are nonzero. This, in addition to the result of [Br] that $d(M)$ vanishes if and only if $[M]$ has a representative fibered over the two-sphere, is enough to show r and d have the same kernel and same range and hence agree. Finally to show $r(M^{4k+3}) = 0$ we use the results of [A-K] to choose a representative of [M] fibered over S^2. On such a representative $r(M)$ is zero by the observation of the previous paragraph.

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