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ABSENCE OF SINGULAR CONTINUOUS SPECTRUM IN N-BODY QUANTUM SYSTEMS[1]

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ABSTRACT. For a large class of potentials including arbitrary bounded potentials with $r^{-2-\epsilon}$ falloff and also allowing suitable local singularities and slower falloff, we demonstrate that the singular continuous spectrum of N-body quantum Hamiltonians is empty. We accomplish this by extending Mourre's work on three body problems to N-bodies.

We want to consider here multiparticle Schrödinger operators, i.e. the Hamiltonian operators of N-body quantum systems. Given a function, V_γ, on R^ν for each pair $\gamma \subset \{1, \ldots, N\}$, the operator

$$\widetilde{H} = -\sum_{i=1}^{N} (2m_i)^{-1} \Delta_i + \sum_\gamma V_\gamma(r_\gamma) \tag{1}$$

on $L^2(R^{N\nu})$ is the Hamiltonian before removal of the center of mass. In (1), we write $r \in R^{\nu N}$ as $(r_1, \ldots, r_N)$; let Δ_i be the Laplacian with respect to r_i and if $\gamma = (ij)$, we write $r_\gamma = r_i - r_j$. If one decomposes $L^2(R^{N\nu}) = \mathcal{H} \otimes \mathcal{H}_{cm}$ with the first factor functions of r_γ and the second functions of $R = (\Sigma m_i)^{-1}(\Sigma m_i r_i)$, then $\widetilde{H} = H \otimes 1 + 1 \otimes T_{cm}$ with $T_{cm} = (2\Sigma m_i)^{-1} \Delta_R$ (see, e.g. [10]). H is the Schrödinger operator we want to discuss. There are three main features of the spectrum of H which one wants to establish in cases where V_γ has suitable falloff at $r_\gamma \longrightarrow \infty$.

(i) Point spectrum can only accumulate at thresholds.

(ii) H has no singular continuous spectrum.

(iii) Scattering is complete.

Thresholds are defined as follows: Let a be a partition of $\{1, \ldots, N\}$ and write $\gamma \subset a$ if γ is a subset of one of the clusters in a. Write $H = H(a) + I(a)$ with $I(a) = \Sigma_{\gamma \not\subset a} V_\gamma$ and write $\mathcal{H} = \mathcal{H}^a \otimes \mathcal{H}_a$ with the first factor functions of r_γ with $\gamma \subset a$ and the second functions of differences of centers of mass of distinct clusters in a. Then $H(a) = H^a \otimes I + I \otimes T_a$: H^a is the Hamiltonian of the internal motion of the clusters and T_a the kinetic energy of motion of the

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clusters relative to each other. Eigenvalues of some H^a with $a \neq a_1$, the one cluster partition, are called thresholds.

In the 1950's the first general results for the simplest case $N = 2$ appeared and during the past twenty years, the case $N = 2$ has been extensively analyzed (see [10], [11] for a review). Faddeev's celebrated work [3] dealt with the solution of (ii) and (iii) in case $N = 3$ but even in that case Faddeev required technical conditions on V_γ roughly requiring $r^{-2-\epsilon}$ falloff and that certain compact operators depending linearly on V_γ not have 1 in their spectrum (there has recently been progress on removing this last restriction [7]). Note that these conditions are now known to imply that the point spectrum has no negative accumulation points, partially solving (i) [2], [15].

There have been solutions of (i)–(iii) for $N \geqslant 4$ assuming strong unproved hypotheses on the various H^a [12]. As for results with assumptions only on the V_γ, we know of three types: (a) solutions of all three problems by Iorio and O'Carroll [5] when the V_γ have small enough norms in suitable topologies; (b) solutions of all three problems by Lavine [6] when all V_γ are repulsive; (c) solutions of problems (i) and (ii) by Balslev and Combes [1] and of (iii) (for "generic" short-range V's in the Balslev and Combes class) by Hagedorn ($N = 4$) [4] and Sigal (all N) [13] when the V_γ have rather strong analyticity properties. In cases (a) and (b), no H^a can have any eigenvalues. Case (c) includes the important special cases of Coulomb and Yukawa potentials. Prior to our work reported here, there were no results depending only on the V's even when all V_γ's were C_0^∞ functions of r_γ.

We consider functions V on R^ν with $V = V_1 + V_2 + V_3$ so that the following six operators are compact after being multiplied by $(1 - \Delta)^{-1}$: (1) $(1 + x^2)V_1$; (2) V_2; (3) V_3; (4) $(1 + x^2)\nabla V_2$; (5) $(1 + x)\nabla V_3$; (6) $(1 + x^2)\nabla\nabla V_3$. Roughly speaking V_1 allows arbitrary potentials with $r^{-2-\epsilon}$ falloff and V_2, V_3 allow slower falloff as long as derivatives falloff sufficiently rapidly. Our main result is

THEOREM (ARBITRARY N). *If all V_γ are of the above form then H has empty singular continuous spectrum and point spectrum can only accumulate at thresholds.*

Our proof is based in part on ideas in a remarkable paper of Eric Mourre [8] preprinted in January, 1979. Mourre focused attention on estimates of the form

$$E_\Delta B E_\Delta \geqslant \alpha E_\Delta^2 + E_\Delta K E_\Delta \tag{2}$$

where E_Δ is a spectral projection for H, K is compact, $\alpha > 0$ and $B = i[H, A]$ with $A = -(i/2)(x \cdot \nabla + \nabla \cdot x)$, the generator of dilations. Mourre requires additional technical hypotheses to which we return shortly and demands (2) hold for Δ, a sufficiently small neighborhood of any nonthreshold points. It is also necessary that the closure of the thresholds be countable: this follows inductively if (i) is known for all subsystems. (2) and the technical hypotheses imply that (i) and (ii) are valid.

Mourre's hypotheses (and ours, also) require that $D(H) = D(H_0)$ ($H_0 = -\Delta$ on $\mathcal{H}$). One then forms the spaces $\mathcal{H}_j = \overline{D(H^{j/2})} = \overline{D(H_0^{j/2})}$ for $j = \pm 2, \pm 1, 0$ (the bar denotes the completion necessary for $j < 0$). Mourre requires that B be bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$ and that $[A, B]$ be bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$. We have improved the required technical hypotheses to only needing that B is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$. In fact our hypotheses on V are precisely chosen to get $[A, V]$ and $[A, [A, V]]$ bounded on the proper spaces. Our improvement here allows nonsmooth V's where Mourre does not.

Mourre also proved (2) for systems with $N = 2$ and $N = 3$ but his method is special to these N. Our main new ideas involve the proof of (2) for general N. The proof is not difficult but is unfortunately complicated. Let us sketch the ideas under the assumption that each H^a has only a finite number of distinct eigenvalues each of them having finite multiplicity (this assumption while not necessary allows considerable simplification in the proof; see [9] for full details). To prove (2) we only need to show that for any $\lambda \notin$ thresholds, there is an f_1 which is identically 1 near λ so that

$$f_1(H)Bf_1(H) \underset{\cdot}{\geqslant} \tfrac{1}{2}\, \mathrm{dist}(\lambda, \text{thresholds})\, f_1(H)^2 \tag{3}$$

where $\underset{\cdot}{\geqslant}$ (and similarly $\underset{\cdot}{=}$) means the inequality (resp. equality) holds, up to a term which is compact plus a term whose norm can be made arbitrarily small by shrinking the support of f_1 suitably.

Our proof requires the notion of *a-compact operator*. Operators, O, commuting with the momentum of relative motion of the clusters in a are called *a-fibered*. Such operators have a direct integral decomposition with respect to the tensor product $\mathcal{H}^a \otimes \mathcal{H}_a$ with fibers $O(p_a)$ acting on $\mathcal{H}^a$. O is called a-compact if it is a-fibered, if the fibers $O(p_a)$ are compact on $\mathcal{H}^a$, and if $p_a \longrightarrow O(p_a)$ is a norm continuous operator vanishing in norm at infinity. A "typical" a-compact operator is $(H_0 + 1)^{-1}P(a)$ where $P(a) = P^a \otimes I$ and P^a = projection onto the point eigenvectors of H^a, which we are assuming is finite rank. For $a = a_1$, we make the convention $P(a_1) = 0$.

a-compact operators have the following properties: (1) they are closed under norm limits; (2) any a-compact operator is a norm limit of a-compact operators with $O(\cdot) \in C_0^\infty$ and with $O(p_a)$ having range and cokernel lying in some p_a-independent finite-dimensional space; (3) if A is a-compact, then, for $a \neq a_1$, $\lim_{|\Delta|\to 0} \|A\overline{P(a)}\, E_\Delta(H(a))\| = 0$, where $\overline{P(a)} = 1 - P(a)$ and $|\Delta|$ = Lebesgue measure of Δ; (4) if $a \subsetneqq b$ and A is b-compact, then $\lim_{|\Delta|\to 0} \|A\, E_\Delta(H_a)\| = 0$; (5) if A is a-compact and B is b-compact, then AB is $a \cup b$ compact; (6) if A is a-compact and B is bounded, a-fibered with continuous fibers, then AB is a-compact.

In the above $a \subset b$ means that the partition a is a refinement of b and the symbol $a \cup b$ is union in the resulting lattice of partitions. Given the approximation property (2), the proofs of the above properties are not difficult.

In [14], Simon proved that

$$f(H) = K^f + \sum_{a_2} j_{a_2} f(H(a_2)) \tag{4}$$

for any continuous f going to zero at infinity. In (4), K^f is a compact operator depending on f, the sum is over all partitions with two clusters and the j's are a suitable partition of unity on $R^{(N-1)\nu}$. In exactly the same way one finds

$$f(H(a_k)) = K^f(a_k) + \sum_{a_{k+1} \subset a_k} j_{a_k, a_{k+1}} f(H(a_{k+1})) \tag{5}$$

where a_k has k-clusters, the sum is over all $k + 1$-cluster partitions and $K^f(a_k)$ is a_k-compact, in fact $K^f(a_k)\,(H_0 + 1)$ is a_k-compact. Given N functions $f_1 \subset f_2 \subset \cdots \subset f_N$ (where $f \subset g$ means $0 \leqslant f \leqslant 1, 0 \leqslant g \leqslant 1$ and $g \equiv 1$ on supp f), we write

$$f_i(H(a_i)) = f_i(H(a_i))\, P(a_i) + f_i(H(a_i))\, \overline{P(a_i)}\, f_{i+1}\,(H(a_i))$$

and then expand the second term using (5). The net result is

$$f_1(H(a_1)) = \sum_{i,a_i} [N(a_i) f_i(H(a_i))\, P(a_i) + M(a_i) f_i(H(a_i))\, \overline{P}(a_i)\, K^{f_{i+1}}(a_i)], \tag{6}$$

where N, M are bounded by combinatorial factors. By successively shrinking the supports of f_N, then $f_{N-1}, \ldots,$ we can be sure that all the $Mf\overline{P}K$ terms have small norm, using property (3) of a-compact operators, or (for $a = a_1$), are compact. For $a \neq b$, $(H_0 + 1)^{-1}P(a)\,P(b)$ is a fixed $a \cup b$-compact operator and either $a \neq a \cup b$ or $b \neq a \cup b$ so the cross terms in f_1^2 have small norm by property (4) if f_N has small support. Thus

$$f_1(H(a_1))^2 \mathrel{\dot{=}} \sum_{i,a_i} N(a_i) f_i(H(a_i))\, P(a_i)^2 f_i(H(a_i))\, N(a_i)^*.$$

Similarly

$$f_1(H(a_1))\, Bf_1(H(a_1)) \mathrel{\dot{=}} \sum_{i,a_i} N(a_i) f(H(a_i))\, P(a_i)\, BP(a_i) f(H(a_i))\, N(a_i)^*.$$

But $B = B^{a_i} \otimes I + 2T(a_i) + \Sigma_{\gamma \not\subset a_i} W_\gamma$ with $W_\gamma = i[V_\gamma, A]$ and $T(c) = I \otimes T_c$. As in Mourre's paper, $f_i P(a_i)\,(B^{a_i} \otimes I)\, P(a_i) f_i = 0$ for f_i having small support (expand $P(a_i)$ into individual projections and control diagonal terms by the Virial theorem and off-diagonal terms by shrinking support). $(H_0 + 1)^{-1}\, P(a_i)\, W_\gamma(H_0 + 1)^{-1}$ is $a_i \cup \gamma$ compact so those terms are small by shrinking f_i. Finally, by shrinking f_i,

$$f_i(H(a_i))\, P(a_i)\, T(a_i)\, P(a_i) f_i(H(a_i)) \geqslant \tfrac{1}{2} \operatorname{dist}(\lambda, \text{thresholds})\, f_i P^2 f_i$$

so (3) results.

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