havior, e.g. regularity and asymptotic behavior. He also has a chapter on nonlinear equations, with emphasis both on the successive approximation theory associated with (7) and on monotone operator methods. (For recent results on equations such as (8), (9) see for instance Kato [1], [2].) The last chapter is concerned with optimal control theory.

Tanabe's book is a good one. A lot of good material is gathered together and unified nicely. Notable features include a nice treatment of fractional powers of operators, a unified exposition of J.-L. Lions' variational approach with evolution operator theory, a sketch of higher order elliptic boundary problems in L^p spaces, $1 < p < \infty$, some nice applications of (nonlinear) monotone operator theory in reflexive Banach spaces, and more.

Unfortunately, the book has some flaws. In many places the English is awkward and there are a number of errors, linguistic, typographical, and mathematical as well.[1] When the book again goes to press, either for a second edition or a new printing, the book will undoubtedly benefit from having the services of a conscientious and competent translation editor.

I wish to thank Professor James G. Hooten of L.S.U. for his helpful comments.

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1. T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, Proc. Sympos. Dundee, 1974, Lecture Notes in Math., no. 448, Springer-Verlag, Berlin and New York, 1975, pp. 25–70.

2. ______, *On the Korteweg-de Vries equation*, Manuscripta Math. **28** (1979), 89–99.

JEROME A. GOLDSTEIN

[1] I'll be happy to supply interested readers with a list of the errors.

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Generalized inverses of linear transformations, by S. L. Campbell and C. D. Meyer, Jr., Surveys and Reference Works in Mathematics, No. 4, Pitman, London, San Francisco and Melbourne, 1979, xi + 272 pp., $42.00.

Although Generalized Inverses (GIs) date back to about 1900, and have been developed more or less continuously since then, with explosive growth since the 1950s (the annotated bibliography of Nashed and Rall [2, pp. 771–1041] lists 1775 related publications through 1975), the subject has long been a somewhat murky backwater. GI theory is notorious for having spawned disproportionately many inferior published articles (presumably hundreds more having been deservedly ambushed on their way to print), and, apart from a wide acceptance by statisticians, there has as yet been only limited interaction with other parts of mathematics. The subject has not penetrated the undergraduate curriculum, and probably most working mathematicians regard GIs as at best a mystery–or even a kind of mysticism.

Nevertheless, certain basic items of GI lore should, in the reviewer's opinion, become part of every mathematician's tool kit; and, among the morass of published work, there are a few articles which contain the beginnings of a substantial, elegant, and useful–though as yet somewhat disconected–theory. Before we examine Campbell and Meyer's book (hereafter referred to as [CM]), it is appropriate to provide a brief outline of what GIs are, and examples of what they can do. In any multiplicative system S with a unity element 1, given any elements $a, x \in S$, then x is called an *inverse* of a if $ax = xa = 1$. In nonassociative systems an element a may have several different inverses $x, y, \ldots,$ but, for associative S, i.e. for *semigroups* (or more specifically, given the presence of 1, monoids), the equation $xa \cdot y = x \cdot ay$ immediately yields the uniqueness of inverses, so that, *when x exists*, we may write $x = a^{-1}$ as a single-valued function. One specific S which is crucial for most GI theory and applications is the multiplicative semigroup $S = M_n(\mathbf{C})$ of all $n \times n$ complex matrices (in which concrete context we shall write A, $X, \ldots,$ rather than $a, x, \ldots$). Here we have the map $A \to A^{-1}$ defined on the "large" subset of nonsingular A, and GI theory originated (after a few historical detours) in the desire to extend this map to obtain a ("natural", well-defined) map $\gamma\colon M_n(\mathbf{C}) \to M_n(\mathbf{C})$, say $A \to A^{\gamma}$.

One way to do this uses the long-known fact that for *every* $A \in M_n(\mathbf{C})$ there exists at least one $X \in M_n(\mathbf{C})$ such that $AXA = A$ (i.e., in the terminology introduced by von Neumann in 1936, every $A \in M_n(\mathbf{C})$ is *regular* in $M_n(\mathbf{C})$, and $M_n(\mathbf{C})$ is also itself called regular). For nonsingular A we are obliged to take $A^{\gamma} = A^{-1}$, but X is nonunique for singular A, so, even if we decide to impose the requirement that $AA^{\gamma}A = A$ for every A, there are infinitely many such "regular extensions" γ; and similar considerations apply to any regular semigroup S (with or without unity) in place of $M_n(\mathbf{C})$.

While, for aesthetic and other reasons, it is desirable to have some "standard" γ available, in fact *any* regular γ yields quick dividends. For example, given any $n \times n$ matrices (or regular semigroup elements) A, B, it is easy to see that the equation $AY = B$ has a solution Y if and only if $AA^{\gamma}B = B$, which leads at once to Penrose's 1955 observation (modestly but erroneously attributed by him to Cecioni) that, given $n \times n$ matrices (or regular ring elements) A, B, C, D, then the equations $AY = B$, $YC = D$ have a common solution Y if (and, of course, only if) each individual equation has a solution and $AD = BC$ (consider $Y = A^{\gamma}B + DC^{\gamma} - A^{\gamma}ADC^{\gamma}$). Moreover, if we regard $n \times n$ matrices as acting on the space V of complex column n-vectors $b, y, z, \ldots,$ then, similarly, $Ay = b$ has a solution $y \in V$ iff $AA^{\gamma}b = b$, in which case the general solution is $y = A^{\gamma}b + (I_n - A^{\gamma}A)z$ for arbitrary $z \in V$, thus avoiding the technicalities of the traditional treatment of linear systems via row transformation and rank. Our restriction here to square matrices is only for ease of exposition–all the ideas generalize trivially to the case of rectangular matrices (and to categories, which provide the corresponding generalization of monoids). There are also already many other significant applications of matrix GIs in diverse special contexts, such as control theory, cryptography, curve fitting, difference and differential equations, electrical network theory, game theory, Markov chains, and programming.

For regular semigroups in general there is (as yet) no known way of canonically singling out a specific regular map $\gamma\colon S \to S$. However, if S has a given *involution* $*$, i.e. a map $*\colon S \to S$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$

for all $a, b \in S$, then a brief computation shows that, for given $a \in S$, there can be at most one $x \in S$ satisfying

$$axa = a, \qquad xax = x, \qquad (ax)^* = ax, \qquad (xa)^* = xa.$$

More surprisingly, subject only to one further condition on the involution, which condition is satisfied by the involution $(a_{rs})^* = (\overline{a_{sr}})$ on $S = M_n(\mathbf{C})$, it is not hard to show that, for regular S, *every* $a \in S$ has a corresponding x. In other words, the four equations displayed just above jointly determine a regular GI map γ: $a \to x$, uniquely defined over the whole of S; this γ is the celebrated Moore-Penrose (MP) map, denoted $a \to x = a^\dagger$. More generally, *any* GI map determined uniquely over all of S (at least for $S = M_n(\mathbf{C})$) will be called a *Unique Generalized Inverse* (UGI).

For certain practical applications, such as solving linear systems $Ay = b$, there may be no advantage in using $A^\dagger$ rather than any other solution X of $AXA = A$; indeed, since *some* such X can be found with substantially less effort than would be needed to compute $A^\dagger$, it may be more efficient to use such a "random" X in preference to $A^\dagger$. However, the MP inverse has many special properties in its favor: for example, it is reflexive (i.e. $(A^\dagger)^\dagger = A$), commutes with * and with unitary similarities, and, if the system $Ay = b$ is inconsistent, then $y = A^\dagger b$ is, in the sense of least squares, the best approximate solution of minimum norm. On the other hand, the MP inverse does have certain shortcomings: e.g. the map $\dagger$ does not commute with some nonunitary similarities, while in general $A^\dagger A \neq AA^\dagger$, $(AB)^\dagger \neq B^\dagger A^\dagger$, and even $(A^2)^\dagger \neq (A^\dagger)^2$. Indeed, *no* single GI or UGI can be expected to satisfy every familiar or desirable property of the ordinary inverse map–there is no "all-powerful" GI. Instead, the best one can hope for is to have available a battery of *several* UGI maps $\gamma_1, \gamma_2, \ldots,$ each with its own characteristic strengths (and weaknesses): for example, Greville's GI is reflexive, while that of the reviewer commutes with * and all similarities and is commuting (i.e. $AA^\gamma = A^\gamma A$), but not regular or reflexive. For any specific application, one employs whichever available UGI (or GI) seems most appropriate to the problem at hand.

Most work on GIs can be roughly classified under the headings (1) invention of UGI maps, (2) exploration of properties of such UGIs, (3) "invention" and properties of nonunique GIs, (4) computational aspects, and (5) applications to other branches of mathematics (i.e. to problems which, as they arise, do not explicitly involve GIs). While, under (3), it is trivial to produce arbitrarily many "definitions" of nonunique GIs, and, under (1), even easy enough to produce new UGIs, research under heading (1) has already passed its most active stage: the easily-produced UGI maps just referred to are artificial and of little interest or utility, and, even for matrices, besides the few UGI maps which have already established themselves, there are probably very few (if any) other *viable* UGI maps left to be discovered.

Thus current research falls largely under (2), . . . , (5), and any text or monograph may usefully be considered in terms of its coverage of these respective headings. While, particularly under (2) and (3), already so much is known that completeness is impracticable, [CM] offers plenty of meat under each heading, and the authors have made an excellent selection of interesting and useful material, providing many new insights from their own wide experience in the field; understandably, they tend to emphasize those topics to which they have themselves made contributions, but this has not led to any serious imbalance. While GI theory can profitably be carried out in more general contexts, certainly most applications to date have used ordinary finite-dimensional complex linear transformations (or, equivalently, complex matrices), so probably the authors are wise in restricting their attention to this concrete context, which is familiar to an extensive audience, provides a ready supply of numerical examples, and also allows the discussion and use of certain valuable concepts and facts which have no analogues in more abstract contexts (for example, the "spectral" property of γ that every eigenvector of A is also an eigenvector of A^γ, or the fact that the bilateral matrix equation $AYC = B$ is equivalent to a system of scalar linear equations for the entries of Y).

The authors provide, in only 272 pages, a thorough treatment of the theory and applications of matrix GIs, mostly accessible to any beginning graduate student or bright undergraduate. There is much stress on motivation and heuristics (see e.g. pp. 19–24), so that the beginner should incidentally absorb some notion of how to create new mathematics himself. Efficient computational algorithms and worked numerical examples (and counterexamples) are liberally provided, and the authors rarely shirk their duties–there is very little hand-waving, and questions raised are pursued relentlessly, the arguments and computations usually being presented in detail in a concrete, down-to-earth (and generally quite readable) informal style. There are numerous well-conceived exercises for the reader.

But some criticisms must be mentioned. Although the authors include several elegant results (some new even to the specialist), this is a utilitarian rather than a stylish book, both in its prose and in its computations. The policy of conscientiously supplying every detail often produces an impression of overkill, which, combined with the inherent grittiness of some of the subject matter (e.g. Chapter 3), may discourage some readers. Further aggravations are careless uses of a single symbol for two meanings (e.g. R in line 3 of p. 35 and line 13 of p. 69, k in lines 3 and 5 of p. 125, or A in lines 3 and 4 of p. 190), other capricious and disconcerting inconsistencies of notation (e.g. $\binom{A\ B}{C\ D}$, $\binom{A\ C}{D\ B}$, and $\binom{A\ C}{R\ D}$ on pp. 53, 58, and 59), and consistently ugly typography: lines are not justified at the right-hand margin and are so close together that superfixes sometimes jostle suffixes from the line above, while insufficient attention has been given to proper display of formulae for maximum intelligibility; and misprints are rather frequent.

It is easy to snipe at any author's choice of material, or at his treatment of it. I content myself with just two such carpings. First, the version of the General Polar Form $A = VB = CW$ offered on p. 73 is much less satisfying than Penrose's 1955 version (in which the imposition of further natural conditions determines B, C, V, W *uniquely*, with $V = W$). Second, the authors conspicuously (pp. 1, 6, 26, 72, 73, 76) shirk establishing the Singular Value Decomposition (and the closely related *canonical* Penrose Decomposition), which is of fundamental importance in matrix MP theory, and (although known for over a century) is rarely accorded even passing mention in linear algebra texts; it would have been well worth taking space to develop this theme at some length.

To summarize, after its renaissance in the 1950s, GI theory passed through its infancy in the 1960s and its adolescence in the 1970s. For the 1980s, it is reasonable to expect a coming-of-age in which abstract algebra, operator theory, and mathematical logic may begin to play a larger role on the theoretical side, while presumably also several significant and interesting new applications remain to be found as GIs become better understood and more widely known. To this end, [CM] deserves a place, together with [1], [2], and [3], on the shelf of every GI specialist and potential GI user (since each source offers much material not treated in the other three), and is also to be recommended to the interested general reader or student. While only a few readers will wish to follow every topic to its last details, this book has enough solid content to make it a valuable reference, and even the beginner should have little difficulty in selecting those sections most deserving of intensive study.

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MICHAEL P. DRAZIN

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Computers and intractability: A guide to the theory of NP-completeness, by Michael R. Garey and David S. Johnson, W. H. Freeman and Company, San Francisco, 1979, xii + 338 pp., $10.00 (paper).

There is a class of algorithmic problems that is currently receiving a great deal of attention from computer scientists and applied mathematicians: the class of "*NP*-complete" problems. Examples of problems in this class are the satisfiability problem for conjunctive normal form statements in the propositional calculus, the three-colorability problem in graph theory, the travelling salesman problem, the three-dimensional matching problem (i.e., the generalization of the classical marriage problem in the setting of three sexes and three-way marriages), the bin packing problem, and the integer programming problem.[1] For each such problem an algorithm is known for solving all instances of the problem; the basis for the monograph reviewed here is the more refined question of whether the problem is tractable, i.e., whether an algorithm exists that solves all instances of the problem and that has running time bounded by a polynomial in the size of the input. (This interpretation of the notion of tractability is due to Cobham [1] and to Edmunds [2].) It is not

[1]At this time it is not known whether the linear programming problem is *NP*-complete, irrespective of the statements in *The New York Times*, November 7, 1979, p. 1.