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Equations of evolution, by Hiroki Tanabe, translated from Japanese by N. Mugibayashi and H. Haneda, Monographs and Studies in Mathematics, No. 6, Pitman, London-San Francisco-Melbourne, 1979, xii + 260 pp., $42.00.

Many mixed problems i.e. initial value-boundary value problems for partial differential equations can be written in the form

$$du(t)/dt = A(u(t)), u(0) = f. \tag{1}$$

Here the unknown function u maps nonnegative time $t \in \mathbf{R}^+ = [0, \infty)$ into a Banach space X, A is an operator acting on its domain $\mathfrak{D}(A) \subset X$ to X, and the initial data f is in $\mathfrak{D}(A)$. The boundary conditions are absorbed into the description of $\mathfrak{D}(A)$, and saying that the solution takes values in $\mathfrak{D}(A)$ amounts to saying that the (time independent) boundary conditions hold for all t. We assume that A is a densely defined linear operator, and we are interested in the case when the problem (1) is well posed, i.e. a solution exists, it is unique, and it depends continuously (in a suitable sense) on the ingredients of the problem, viz. f and A. When this is the case let $T(t)$ map the solution at time 0 (i.e. f) to the solution at time t (i.e. $u(t)$). Then the uniqueness gives the semigroup property $T(t)T(s) = T(t + s)$ for $t, s \in \mathbf{R}^+$, and we have $T(t) =$ "e^{tA}" at least formally; but in general A is an unbounded operator so one must be careful.

The Hille-Yosida-Phillips theory of (one parameter strongly continuous) semigroups of (linear) operators makes this all precise. The theory says that (1) is well posed iff it is governed by a semigroup $T = \{T(t): t \in \mathbf{R}^+\}$ iff A generates a semigroup T; and moreover, A generates a semigroup T iff A satisfies certain explicitly verifiable conditions. For instance, when the semigroup is contractive i.e. $\|T(t)\| < 1$ for all $t > 0$, the exponential formula

$$T(t)f = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} f$$

suggests that T can be recovered from A if $(I - \lambda A)^{-1}$ is an everywhere defined contraction (i.e. $\|(I - \lambda A)^{-1}\| < 1$) for each $\lambda > 0$. In this case A is called m-dissipative, and this condition is both necessary and sufficient for A to generate a contraction semigroup.

More generally, solving (1) by semigroup methods reduces to solving the time-independent problem

$$u - \lambda Au = h \tag{2}$$

for u given an arbitrary $h \in X$ and a sufficiently small $\lambda > 0$, and showing that the u satisfies certain conditions involving norm inequalities (such as $\|u\| \leq \|h\|$). When (1) is a parabolic mixed problem, (2) becomes an elliptic boundary-value problem. But (1) also includes hyperbolic problems and other types of problems as well.

The theory of semigroups of operators has been extensively developed, and the applications of this theory have greatly increased our understanding of mixed problems for partial differential equations.

On the other hand, one wants to consider equations where the coefficients of the differential operator and/or the boundary conditions depend on time. This means that the generator A in (1) should be time dependent: $A = A(t)$. In this case (1) is called a temporally inhomogeneous equation or an equation of evolution (although the solution u evolves in time in any case, so that (1) should be called an evolution equation whether or not A is time dependent). Write the solution of the well-posed problem

$$du(t)/dt = A(t)(u(t)), u(s) = f \tag{3}$$

as $u(t) = U(t, s)f$; the analogue of the semigroup property is

$$U(t, s)U(s, r) = U(t, r)$$

for $t \geq s \geq r$. The operator family $U = \{U(t, s)\colon t \geq s \geq 0\}$ has many aliases; it is called an evolution family, an evolution operator, a fundamental solution, a propagator, etc.

Here is a formal construction of U. We want to solve (3) for $u(t)$. Partition $[s, t]$ as $s = \tau_0 < \tau_1 < \cdots < \tau_n = t$ and choose $\tilde{\tau}_i \in [\tau_{i-1}, \tau_i]$. If $A(\tilde{\tau}_i)$ approximates $A(r)$ for $\tau_{i-1} \leq r \leq \tau_i$, then

$$v(t) = T_n(\tau_n - \tau_{n-1}) \cdots T_2(\tau_2 - \tau_1)T_1(\tau_1 - \tau_0)f \tag{4}$$

approximates $u(t)$ as $\max_i(\tau_i - \tau_{i-1}) \to 0$; here T_i denotes the semigroup generated by $A(\tau_i)$. Since the operators appearing in (4) do not commute in general, the order is important. The expression (4) is just the Cauchy-Peano polygonal approximation in operator-theoretic form: $v(\tau_1) = T_1(\tau_1 - \tau_0)f$ is the approximate solution at τ_1, $v(\tau_2) = T_2(\tau_2 - \tau_1)v(\tau_1)$ is the approximate solution at τ_2, etc.

The solution of the inhomogeneous problem

$$du(t)/dt = A(t)u(t) + g(t), u(s) = f \tag{5}$$

is given by the variations of parameters formula

$$u(t) = U(t, s)f + \int_v^t U(t, r)g(r)\, dr. \tag{6}$$

When A is independent of t and $s = 0$, (6) reduces to

$$u(t) = T(t)f + \int_0^t T(t - r)g(r)\, dr.$$

The nonlinear problem

$$du(t)/dt = A(t)u(t) + g(t, u(t)), u(s) = f \tag{7}$$

can often be solved by successive approximations:

$$u_n(t) = U(t, s)f + \int_s^t U(t, s)g(r, u_{n-1}(r))\, dr.$$

This comes from making a guess $u_1(t)$, plugging it into the nonlinear (g) term in (7), solving the resulting inhomogeneous equation by (6), and iterating. In favorable circumstances, a fixed point theorem can be used to show that u_n converges (at least locally in time) to a solution of (7).

Another nonlinear equation that can be solved by iteration (in some cases) is

$$du(t)/dt = A(t, u(t))u(t), u(s) = f. \tag{8}$$

Here $A(t, w)$ should be a generator for which (t, w). The approximate solution solves

$$du_n(t)/dt = A(t, u_{n-1}(t))u_n(t), u_n(s) = f. \tag{9}$$

Even when A does not depend on t explicitly, so that (8) is $du/dt = A(u)u$, (9) becomes

$$du_n(t) = A(u_{n-1}(t))u_n(t) \equiv A_n(t)u_n(t),$$

which is a temporally inhomogeneous linear equation. An example is the Korteweg-de Vries equation

$$\partial u/\partial t = \partial^3 u/\partial x^3 + u\partial u/\partial x.$$

Here X is a space of functions of $x \in \mathbf{R}$, and one can take

$$A(w)v = d^3v/dx^3 + wdv/dx$$

(or $A(w)v = d^3v/dx^3 + vdw/dx$). Thus the nonautonomous linear equation (3) is important not only for linear problems but also for nonlinear autonomous problems as well.

Before discussing Tanabe's book, let me recall briefly some important names in the history of temporally inhomogeneous equations of evolution. The first major result for (3) was obtained by T. Kato in 1953. Definitive results for parabolic versions of (3) (involving analytic semigroups) were established around 1960 and thereafter by Kato, H. Tanabe, and P. Sobolevskiĭ. Definitive results for (3) is general, including hyperbolic versions, were obtained in the early seventies by Kato. Many other authors have made important contributions, but two points are worth emphasizing. First, Kato is *the* leader in this field; it is hard to overemphasize the importance of his work. Second, Tanabe is an international authority in this field; it seems natural for him to have written the book under review.

Tanabe's book covers semigroups of linear operators and temporary inhomogeneous equations of evolution. It should be accessible to graduate students having some background in functional analysis and partial differential equations (although the exposition becomes quite sketchy in places). Applications include parabolic equations and symmetric hyperbolic systems. In addition to well-posedness results, Tanabe gives results on qualitative behavior, e.g. regularity and asymptotic behavior. He also has a chapter on nonlinear equations, with emphasis both on the successive approximation theory associated with (7) and on monotone operator methods. (For recent results on equations such as (8), (9) see for instance Kato [1], [2].) The last chapter is concerned with optimal control theory.

Tanabe's book is a good one. A lot of good material is gathered together and unified nicely. Notable features include a nice treatment of fractional powers of operators, a unified exposition of J.-L. Lions' variational approach with evolution operator theory, a sketch of higher order elliptic boundary problems in L^p spaces, $1 < p < \infty$, some nice applications of (nonlinear) monotone operator theory in reflexive Banach spaces, and more.

Unfortunately, the book has some flaws. In many places the English is awkward and there are a number of errors, linguistic, typographical, and mathematical as well.[1] When the book again goes to press, either for a second edition or a new printing, the book will undoubtedly benefit from having the services of a conscientious and competent translation editor.

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[1]I'll be happy to supply interested readers with a list of the errors.

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Generalized inverses of linear transformations, by S. L. Campbell and C. D. Meyer, Jr., Surveys and Reference Works in Mathematics, No. 4, Pitman, London, San Francisco and Melbourne, 1979, xi + 272 pp., $42.00.

Although Generalized Inverses (GIs) date back to about 1900, and have been developed more or less continuously since then, with explosive growth since the 1950s (the annotated bibliography of Nashed and Rall [2, pp. 771–1041] lists 1775 related publications through 1975), the subject has long been a somewhat murky backwater. GI .theory is notorious for having spawned disproportionately many inferior published articles (presumably hundreds more having been deservedly ambushed on their way to print), and, apart from a wide acceptance by statisticians, there has as yet been only limited interaction with other parts of mathematics. The subject has not penetrated the undergraduate curriculum, and probably most working mathematicians regard GIs as at best a mystery–or even a kind of mysticism.

Nevertheless, certain basic items of GI lore should, in the reviewer's opinion, become part of every mathematician's tool kit; and, among the