you are immersed, though perhaps only up to your knees, in the sea of categories and functors. Similar remarks apply to the definition of operator ideals, which takes you in up to your waist, and then thinking about the relation between spaces of linear maps, summable operators, quotients, subspaces and duals–well perhaps one should learn to swim after all.

For the novice there is of course a big difference between swimming in your depth where, if in trouble, you can revert to a more familiar attitude, and moving into deeper water. In the book under review the change occurs about 2/3 of the way through, at the beginning of Chapter 4. The first three chapters cover familiar material about Banach spaces, tensor products and modules over Banach algebras though there are categorical overtones. The remainder deals with material which will be new to many functional analysts. Just as in Banach algebra theory the basic object of study is the algebra of bounded operators on a Banach space and its closed subalgebras, in Banach category theory the basic object is some set of Banach spaces and the bounded maps between them. Obviously the composition product of two such maps is not always defined but algebraic category theory is designed to cope with exactly this problem. Representations of Banach algebras, that is Banach modules, from the more familiar theory are replaced by functors from the Banach category into the category of all Banach spaces. Notions such as tensor products extend to the more general situation too.

One advantage of discussing a subject so ill thought of as "general nonsense" is that it doesn't need much to show that the reputation is undeserved and the authors certainly achieve this and more. On the other hand it is fair to say that the subject is mainly descriptive and does not solve any pre-existing problems–as did the Banach algebra approach to Wiener's theorem for example. For anyone curious to see how category theory can be applied to the basic structures of functional analysis this is a lively introduction with plenty of applications to concrete situations. The reader unfamiliar with the basic notions of category theory (e.g. category, small category, full subcategory) will need to familiarize himself with them from another source as they are not explained here–an unfortunate blemish but not a fatal flaw.

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GO-spaces and generalizations of metrizability, by J. M. van Wouwe, Mathematical Centre Tracts, Volume 104, Mathematisch Centrum, Amsterdam, The Netherlands, 1979, x + 117 pp.

The general topology of product spaces is a theory which bristles with counterexamples. Think of a good question concerning normality, paracompactness or the Lindelöf property in product spaces, and chances are that an example, rather than a theorem, will settle the matter. Many of the most durable of these examples are built using lines of various kinds. Must the product of two paracompact spaces be paracompact? No, consider the square of the Sorgenfrey line–it's not even normal [S]. Must the product of a paracompact space and a separable metric space be paracompact? No, consider the product of the Michael line with the usual space of irrational numbers–it's also not even normal [$\mathbf{Mi_1}$]. Must the product of two paracompact spaces be paracompact provided it *is* normal? Consistently no, since by mixing in a little modern set theory (specifically, Martin's Axiom plus the negation of the Continuum Hypothesis) Przymusiński found a subspace of the Sorgenfrey line whose square is normal but not paracompact [P]. And in that same set theoretic setting, Nyikos [Ny] used another kind of line to settle a question of Katětov, exhibiting a compact, nonmetrizable Hausdorff space X whose square is hereditarily normal. (Katětov had observed that if X^3 is hereditarily normal for a compact space X, then X must be metrizable.) Further examples of the uses of lines in product theory appear in [Al] and [$\mathbf{Mi_2}$].

All of the lines mentioned above arise from the following general construction. Beginning with a linearly ordered set $(X, <)$, select four disjoint sets R, E, I, and L which cover X. Construct a topology for X by: (1) isolating all points of I; (2) using all intervals of the form $[x, b)$ as a neighborhood base at x in case $x \in R$; (3) using all intervals of the form $(a, x]$ as a neighborhood base at x in case $x \in L$; and (4) using all intervals of the form (a, b), where $a < x < b$, as a neighborhood base at x in case $x \in E$. The resulting space is denoted by $GO(R, E, I, L)$ and is called a *generalized ordered space*. To obtain the Sorgenfrey line, let $X = \mathbf{R}$, the usual set of real numbers, and form $GO(\mathbf{R}, \varnothing, \varnothing, \varnothing)$. To obtain the Michael line, construct $GO(\varnothing, \mathbf{Q}, \mathbf{P}, \varnothing)$ where $\mathbf{Q}$ and $\mathbf{P}$ are the sets of rational and irrational numbers respectively.

It seems that Čech introduced generalized ordered spaces in his seminar during the 1930's. Since the earliest days of this century, mathematicians had been studying *linearly ordered topological spaces*, which are obtained by equipping linearly ordered sets with the usual open-interval topology. It is easy to see that if a linearly ordered set $(X, <)$ is equipped with the open-interval topology and if $Y \subset X$, then the relative topology which Y inherits from X need not coincide with the open-interval topology induced on Y by the restriction of $<$ to Y. Čech observed that the generalized ordered spaces are precisely those spaces which can be topologically embedded in a linearly ordered space, and that the generalized ordered class is a hereditary class.

A generalized ordered space gets its structure from the sets R, E, I, and L used in its construction, of course, but also from the linear ordering of the underlying set which can itself be quite complex. The most familiar linearly ordered sets–the set $\mathbf{R}$ and the set $[0, \omega_1)$ of countable ordinals–have orderings which are uncharacteristically simple. More complex orderings can arise from lexicographic products. For example, with $\mathbf{P}$, $\mathbf{Q}$ and ω_1 as above, and $0 < \lambda < \omega_1$, consider the set $X(\lambda)$ consisting of all functions t: $[0, \lambda] \to \mathbf{R}$ which have $t(\alpha) \in Q$ if $\alpha < \lambda$ and $t(\lambda) \in \mathbf{P}$. Let $X = \bigcup \{X(\lambda)\colon \lambda < \omega_1$ is a limit ordinal$\}$. Order X lexicographically, i.e., if $s \neq t$ belong to X, let α be the first ordinal where $s(\alpha) \neq t(\alpha)$ and define $s \prec t$ if $s(\alpha) < t(\alpha)$ in the usual ordering of $\mathbf{R}$ [Be]. Another interesting linearly ordered set which can arise from a lexicographic product is an η_1-set, i.e., a linearly ordered set X with the property that if A and B are countable subsets of X which satisfy $a < b$

for each $a \in A$ and $b \in B$, then there are points $p, q \in X$ having $a < p < q < b$ for each $a \in A$ and each $b \in B$ [GJ]. Much further from the real world, one might find linear orderings which yield Souslin lines (linearly ordered spaces which are not separable and yet in which every disjoint family of open intervals is countable), but whether or not such things exist depends on your set theory [$\mathbf{R}_2$].

Given a topological space, how can one recognize whether its topology can come from a linear ordering? Eilenberg [E] characterized connected, locally connected spaces whose topology is the open-interval topology of some linear ordering: they are precisely the spaces X for which the set $\{(x, y): x \neq y\}$ is a disconnected subspace of $X \times X$. Rudin [$\mathbf{R}_1$] answered the question "When is a subspace of $\mathbf{R}$ orderable (perhaps by some other ordering than the one inherited from $\mathbf{R}$)?" and announced a general solution to the question "When can the topology of a generalized ordered space be the open-interval topology of some linear ordering?" Herrlich [H] characterized metric spaces which are orderable. And finally, van Dalen and Wattel [vDW] solved the general orderability problem by proving (1) that a Hausdorff space is a generalized ordered space if and only if it has a subbase $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$ where each $\mathbb{S}_i$ is linearly ordered by inclusion and (2) that a Hausdorff space is linearly orderable if and only if it has a subbase $\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2$ where each $\mathbb{S}_i$ is linearly ordered by set-inclusion and has the property that if $T \in \mathbb{S}_i$ satisfies $T = \bigcap \{S \in \mathbb{S}_i: T \subset S, T \neq S\}$ then T must also satisfy

$$T = \bigcup \{S \in \mathbb{S}_i: S \subset T, S \neq T\}.$$

Generalized ordered spaces are well behaved from the viewpoint of general topology. Three examples of this good behavior concern Dowker spaces, monotonic normality and the Dugundgi extension theorem. Any generalized ordered space has the property that its product with the closed unit interval is normal, so there are no generalized ordered Dowker spaces [Ba]. Second, every generalized ordered space is monotonically normal, i.e., for each pair (A, U) where A is closed and U is open and $A \subset U$, there is an open set $G(A, U)$ satisfying $A \subset G(A, U) \subset \mathrm{cl}(G(A, U)) \subset U$ and satisfying $G(A, U) \subset G(B, V)$ whenever $A \subset B$ and $U \subset V$ [HLZ]. Third, every generalized ordered space satisfies a strong form of the Dugundji extension theorem, i.e., if A is a closed subspace of a generalized ordered space X, then there is a linear, norm-preserving extender $e: C^*(A) \to C^*(X)$, where $C^*(A)$ and $C^*(X)$ denote the Banach spaces of continuous, bounded, real-valued functions of A and X, respectively, equipped with sup-norm [HL]. (All of these normality results use the Axiom of Choice in an obvious way and an old problem of Birkhoff [Bi] asks whether linearly ordered spaces are provably normal without that axiom.)

The theory of generalized ordered spaces becomes less predictable when one turns from normality to metrization theory. The basic metrization theorem for linearly ordered spaces asserts that a linearly ordered space is metrizable if and only if the diagonal $\{(x, x): x \in X\}$ is a G_δ-subset of the product space $X \times X$ [L]. That result is definitely false in the wider class of generalized ordered spaces since one can easily observe that the Sorgenfrey line is not metrizable and yet has a G_δ-diagonal. (That observation leads to an even more interesting fact: there is no linear ordering of $\mathbf{R}$ which yields the Sorgenfrey line as its open-interval topology.) The proper version of the G_δ-diagonal theorem for generalized ordered spaces was given by Faber in [Fa]: a generalized ordered space X is metrizable if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of X satisfying (1) for each $p \in X$, $\bigcap\{\mathrm{St}(p, \mathcal{G}_n): n \geq 1\} = \{p\}$, where $\mathrm{St}(p, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n: p \in G\}$, and (2) the set $\{p \in X: \{\mathrm{St}(p, \mathcal{G}_n): n \geq 1\}$ is not a neighborhood base at $p\}$ is a σ-discrete subset of X. Faber's result takes care of about half of the metrization theory for generalized ordered spaces. The monograph under review here studies much of the rest.

Van Wouwe's monograph concentrates on generalized metric spaces called p, M and Σ-spaces, and perhaps a preliminary word about these types of topological spaces is in order here. We say that a completely regular space X is a p-space [A] if there are collections $\mathcal{H}_1, \mathcal{H}_2, \ldots$ of open subsets of the Čech-Stone compactification of X such that for each $p \in X$, $\varnothing \neq \bigcap\{\mathrm{St}(p, \mathcal{H}_n): n \geq 1\} \subset X$. We say that a space X is an M-space [Mo] if there is a sequence $\mathcal{J}_1, \mathcal{J}_2, \ldots$ of open covers of X such that $\mathcal{J}_{n+1}$ star-refines $\mathcal{J}_n$ and such that if $p \in X$ and $x_n \in \mathrm{St}(p, \mathcal{J}_n)$ for each $n \geq 1$, then the sequence $\langle x_n \rangle$ has at least one cluster point in X. Finally, we say that a space X is a Σ-space [Na] if there is a sequence of locally finite closed covers $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of X such that for each $p \in X$, the set

$$C(p) = \bigcap\{F\text{: for some } n \geq 1, p \in F \in \mathcal{F}_n\}$$

is countably compact and has the property that if U is open and contains $C(p)$, then for some $n \geq 1$, $\bigcap\{F \in \mathcal{F}_n: p \in F\} \subset U$. In what sense might these three very technical-sounding types of spaces be called generalized metric spaces? First, every metric space is p, M and Σ. Second, each of the properties defining p, M and Σ-spaces is a "factor" of metrizability in a nontrivial and useful theorem. For example, a space is metrizable if and only if it is paracompact, a p-space, and has a G_δ-diagonal [Bo], [O], and a space is metrizable if and only if it is an M-space having a point-countable base [F], [Mc]. Third, while many nonmetrizable spaces are p, M or Σ-spaces (any locally compact group is all three, and any CW-complex is a Σ-space), spaces which are p, M or Σ have enough structure to guarantee certain metric-like behavior, e.g., that the product of countably many spaces, each of which is paracompact and p, M or Σ, must be paracompact.

In general, there is no relationship between p-spaces and M-spaces. The first major result in van Wouwe's monograph establishes that in the class of generalized ordered spaces, p implies M. (The same result for linearly ordered spaces has been independently obtained by Velichko [V].) The proof is natural and elegant. Given any generalized ordered space X, van Wouwe constructs two quotient spaces gX and cX of X, both of which are again generalized ordered spaces, and a natural closed continuous mapping from gX onto cX. He then proves that X is a p-space if and only if gX is metrizable, and that X is an M-space if and only if cX is metrizable. Hence, if X is a p-space, then cX is seen to be the image of a metrizable space under a closed, continuous mapping. While closed mappings cannot, in general, be counted on to preserve metrizability, they always preserve a weaker structure called a semistratification which, for generalized ordered spaces, is known to be equivalent to metrizability. Thus, if X is a p-space, then cX is metrizable so that X must be an M-space.

A second section of van Wouwe's monograph gives new proofs of a curious known result: a generalized ordered space is metrizable if and only if every subspace of it is either p or M [BeL]. The final section of the monograph attacks the analogous problem for Σ-spaces: must a generalized ordered space be metrizable if each of its subspaces is known to be a Σ-space? To date, this problem remains unsolved, even if one restricts attention to compact spaces. Van Wouwe obtains several interesting reductions of the problem. For example, it would solve the general problem if someone would prove that every closed subset of a Lindelöf generalized ordered space X must be a G_δ-set, given that every subspace of X is a Σ-space.

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Equations of evolution, by Hiroki Tanabe, translated from Japanese by N. Mugibayashi and H. Haneda, Monographs and Studies in Mathematics, No. 6, Pitman, London-San Francisco-Melbourne, 1979, xii + 260 pp., $42.00.

Many mixed problems i.e. initial value-boundary value problems for partial differential equations can be written in the form

$$du(t)/dt = A(u(t)), u(0) = f. \tag{1}$$

Here the unknown function u maps nonnegative time $t \in \mathbf{R}^+ = [0, \infty)$ into a Banach space X, A is an operator acting on its domain $\mathfrak{D}(A) \subset X$ to X, and the initial data f is in $\mathfrak{D}(A)$. The boundary conditions are absorbed into the description of $\mathfrak{D}(A)$, and saying that the solution takes values in $\mathfrak{D}(A)$ amounts to saying that the (time independent) boundary conditions hold for all t. We assume that A is a densely defined linear operator, and we are interested in the case when the problem (1) is well posed, i.e. a solution exists, it is unique, and it depends continuously (in a suitable sense) on the ingredients of the problem, viz. f and A. When this is the case let $T(t)$ map the solution at time 0 (i.e. f) to the solution at time t (i.e. $u(t)$). Then the uniqueness gives the semigroup property $T(t)T(s) = T(t + s)$ for $t, s \in \mathbf{R}^+$, and we have $T(t) =$ "e^{tA}" at least formally; but in general A is an unbounded operator so one must be careful.

The Hille-Yosida-Phillips theory of (one parameter strongly continuous) semigroups of (linear) operators makes this all precise. The theory says that (1) is well posed iff it is governed by a semigroup $T = \{T(t): t \in \mathbf{R}^+\}$ iff A generates a semigroup T; and moreover, A generates a semigroup T iff A satisfies certain explicitly verifiable conditions. For instance, when the semigroup is contractive i.e. $\|T(t)\| < 1$ for all $t > 0$, the exponential formula

$$T(t)f = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} f$$

suggests that T can be recovered from A if $(I - \lambda A)^{-1}$ is an everywhere defined contraction (i.e. $\|(I - \lambda A)^{-1}\| < 1$) for each $\lambda > 0$. In this case A is called m-dissipative, and this condition is both necessary and sufficient for A to generate a contraction semigroup.