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Analysis, manifolds and physics, by Yvonne Choquet-Bruhat, Cecile de Witt-Morette, and Margaret Dillard-Bleick, North-Holland, Amsterdam, The Netherlands, 1977, xviii + 544 pp., $19.50.

Physical mathematics has always been an important part of mathematics as a discipline which concerns itself with deepening and uncovering mathematical theorems by interpreting them in the light of applications to physics. Of course, in mathematics one is often faced with the challenge of putting a result in the right perspective ("what does this really mean?"), to look at it the right way; but even more so when it comes to relating the formulas to the "real world". Many an apology has been made on behalf of the cult of pure mathematics (pure almost in the sense of virgin, untouched by any reality but the mathematical), that here is where the beauty of the subject is found. This point of view is in turn still under fire from those advocating less abstraction and more solution in mathematics. I think there is an in-between, indeed I see a genuine interest in the mathematical community in applications of mathematics, in combining abstract beauty with concrete power, and even remote hopes of assisting physics in its many struggles with fields and particles.

Theoretical physics deals with building models of so-called physical systems; speaking of a physical system already breaks down the universe in two parts: the system plus a background (to the neglected or influencing the system in a given way). This jig-saw puzzle approach must add up to our given universe (the only true physical system: "les lois physiques concernent tous les mondes possibles, alors que le monde réel n'est tiré qu'à un seul exemplaire" (H. Poincaré))–a complicated verification by experimental physics.

Perhaps the system to which most attention (and success) has been devoted is that of the Hydrogen atom: a point particle moving in $\mathbf{R}^3$ under the influence of a central force field with potential $-r^{-1}$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Classically, each instant the system is in some (pure) state of position $\mathbf{x} = (x_1, x_2, x_3)$ and linear momentum $\mathbf{p} = (p_1, p_2, p_3)$.

$$M = \{(\mathbf{x}, \mathbf{p})|\mathbf{x}, \mathbf{p} \in \mathbf{R}^3, \mathbf{x} \neq \mathbf{0}\}$$

is called the phase space. The observables are functions on M such as

$$h = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - r^{-1} \quad \text{(energy)}$$

$$\mathbf{l} = \mathbf{x} \times \mathbf{p} \quad \text{(angular momentum)}$$

$$\mathbf{a} = \mathbf{l} \times \mathbf{p} + r^{-1}\mathbf{x} \quad \text{(Runge-Lenz vector).}$$

Relative to the Poisson parenthesis on M

$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} \right)$$

we see that

$$\{h, l_i\} = \{h, a_i\} = \{l_i, a_j\} = 0$$

$$\{l_1, l_2\} = -l_3, \{l_1, a_2\} = -a_3, \{a_1, a_2\} = 2hl_3 \tag{1}$$

where the last relations are cyclic in 1, 2, 3. The time-evolution of any observable $f = f(\mathbf{x}(t), \mathbf{p}(t))$ is given by

$$\frac{df}{dt} = \{h, f\}.$$

In particular for $\mathbf{x}(t)$ this says that (for mass one)

$$\frac{d^2}{dt^2}\mathbf{x}(t) = -r(t)^{-3}\mathbf{x}(t) \tag{2}$$

and it amounts to saying that the system evolves in time on M along the Hamiltonian vector field $\{h, \cdot\}$. This is a typical classical Hamiltonian system with what seems to be an overly elaborate structure–after all, aren't we just supposed to solve (2)? Not quite, we are also interested in the symmetries of the system, and (1) tells us that $\mathbf{l}$ and $\mathbf{a}$ are preserved under the motion (a plane orbit with no precession). Also $\mathbf{y} = \mathbf{l} + \mathbf{a}$ and $\mathbf{z} = \mathbf{l} - \mathbf{a}$ restricted to the level surface $h = -\frac{1}{2}$ generate (under Poisson parenthesis) two mutually commuting Lie algebras of $SO(3)$, and the same functions map the manifold M_E of orbits in M of energy $-\frac{1}{2}$ one-to-one onto a direct product of 2 Riemannian spheres, so $M_E = S^2 \times S^2$. Thus the Hydrogen atom possesses not just the obvious $SO(3)$-symmetry but actually an $SO(3) \times SO(3)$-symmetry, which was not apparent just by looking at (2). In the quantum mechanical description of the system, we encounter the same observables, this time as selfadjoint linear operators on $H = L^2(\mathbf{R}^3)$ (the state space of Schrödinger wave functions), and also the same symmetries, replacing the Poisson bracket by the commutator between linear operators. The negative energy spectrum (bound states) is $\{-n^{-2}|n = 1, 2, 3, \ldots\}$, the group $SU(2) \times SU(2)$ acts on H_B, the Hilbert space of bound states, and there is an essentially unique homomorphism from the semisimple Lie group $SU(2, 2)$ into the group of unitary operators on H_B with the given action of tne subgroup $U(1) \times SU(2) \times SU(2)$ (here $U(1)$ gives the one-parameter group corresponding to the energy). It is a curious coincidence that this unitary representation of $SU(2, 2)$ is equivalent to the representation on the space of positive-frequency solutions to the wave equation on $\mathbf{R}^3 \times \mathbf{R}$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)\varphi = 0$$

(or the analogous equation on $\mathbf{R} \times S^3$). This is also in a certain sense the smallest unitary representation of $SU(2, 2)$ (much like the defining representation of say $SU(2)$ is for that group).

The previous example (which forgot about the quantum structure of photons and electrons) touches (although lightly) on several of the mathematical theories that we are to consider in connection with analysis, manifolds and physics.

The title of the book under review invites many speculations; it espouses a good part of the library and of current research. Which then, are the areas of analysis, the theory of manifolds, and physics which have something in common? What should be included in a (long and difficult) hypothetical graduate course taught under the same heading?

From analysis I would first choose general integration theory **[18]** plus harmonic analysis on Lie groups (already an ambitious step indeed, but this is a good class)–mostly to account for representations of symmetry groups in quantum mechanics **[14]**, **[5]**, **[23]** and to get a close look at special functions. Then on to differential geometry **[8]**, **[6]**, not just to encompass space-times but more importantly with bundles, connections and complex manifolds in mind. Consider as an example the description of a Yang-Mills gauge field on an open set U in the flat space-time $\mathbf{R}^3 \times \mathbf{R}$: let $f(x)$ be a function (a physical field) on U with values in a complex vector space V (internal symmetry space) carrying a unitary representation of the (gauge) Lie group G. $f(x)$ represents the field from entities such as mesons or quarks and V provides room for internal degrees of freedom of such particles (e.g. isospin for $G = SU(2)$). G acts in a natural way on $f(x)$ viz. for $g \in G$ and π the representation in question

$$f(x) \to \pi(g)f(x). \tag{3}$$

This is called a global gauge transformation as opposed to the local gauge transformation

$$f(x) \to \pi(g(x))f(x) \tag{4}$$

where g now may depend on x. The minimal invariant subspaces of V are thought of as the various types or families of particles making up the field $f(x)$ (say families of different isospin).

Starting (as is customary) from a Lagrangian density

$$L = L(f(x), (\nabla f)(x))$$

where ∇ denotes the gradient on U, assumed to be invariant under all transformations (3), one wishes to modify L to become invariant under everything of the form (4). The answer is to replace the straight differentiation ∇ by a covariant differentiation

$$D = \nabla + \pi(a(x)) \tag{5}$$

so that

$$\tilde{L} = L(f(x), (Df)(x)).$$

Here $a(x)$ has 4 components, each taking values in the Lie algebra of G, and we again by π denote the action of the Lie algebra on V. The rule (5) is often in physics referred to as a minimal coupling, and letting $U(x) = \pi(g(x))$ and $A(x) = \pi(a(x))$, while (4) takes place $A(x)$ transform like

$$\begin{aligned} A(x) &\to U(x)A(x)U(x)^{-1} + U(x)\nabla(U(x)^{-1}) \\ &= \pi(\mathrm{Ad}_G(g(x)) \cdot a(x)) + U(x)\nabla(U(x)^{-1}). \end{aligned} \tag{6}$$

Ad_G is the adjoint action of G on its Lie algebra. That $\tilde{L}$ is indeed invariant under (4) and (6) follows easily from the differentiation formula

$$\begin{aligned} 0 &= \nabla(U(x)U(x)^{-1}) \\ &= (\nabla U(x))U(x)^{-1} + U(x)\nabla(U(x)^{-1}). \end{aligned}$$

From the requirement of local gauge invariance we have thus arrived at $\tilde{L}$, which has in it the extra (gauge) field $A(x)$, an analogue of the vector potential in electromagnetism. The nonlinear system of differential equations to be satisfied by $f(x)$ and $A(x)$ are now the Euler-Lagrange equations for stationary points of the action

$$\int_U (\tilde{L} + F^2)$$

where F^2 is a certain canonical gauge-invariant quadratic expression in the covariant derivatives of $A(x)$. $L = \tilde{L} = 0$ corresponds to the Yang-Mills equations in vacuum, for which many special (in particular so-called selfdual) solutions have been found. In the case of $G = SU(2) \times U(1)$ (unifying weak and electromagnetic interactions) one is still trying to identify in huge accelerator experiments members (W-mesons: one of the predictions of the recent Nobel laureates) of the (quantized) field $A(x)$.

It is remarkable, that a physical invariance leads us to consider one of the basic structures in modern geometry: that of a fiber bundle and associated vector bundles (in this case over space-time) with connection ($a(x)$ represents a one-form), internal symmetries being mirrored in the geometry of the fiber group G, and the location in space and time in the differential geometry on the base.

The recent success of the theory of twistors [4] in translating self-dual connections on the four-sphere to certain rank 2 vector bundles on complex projective 3-space may be viewed as an instance of the integral geometry of a double fibration of a manifold Z:

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y. \end{array}$$

Here X could be $\mathbf{R}^n$ and Y equal to $\mathbf{R}^+ \times S^{n-1}$, thought of as the manifold of all hyperplanes in X (think of a hyperplane tangent to a sphere around the origin). Now take Z to be the subset of $X \times Y$ for which $x \in y$–the diagram then is related to the Radon transform (to a function on X associate its integrals over all possible hyperplanes). As in deformation theory for complex manifolds and as for the Radon transform, points in X parametrize certain submanifolds of Y and vice versa. Via Z then, geometric objects may be lifted between X and Y [27], [7], [11]. Poisson-parentheses as in the example with the Hydrogen atom are part of the discipline called symplectic geometry. Perhaps the central issue will be symplectic geometry and the behavior of Lagrangian submanifolds, which provide the machinery of classical field and particle dynamics [2], [1], [9], and literally expose the functor of passing from classical to quantum mechanics [26], [20]. Connected with this is the propagation of singularities in partial differential equations as in geometrical optics or the Dirichlet and Neumann problems for the wave equation [22]. One aspect of analysis not to be forgotten is spectral theory, expansions in eigenfunctions and spectral kernels, which provide the mathematics of nonbound eigenstates of quantum systems. For instance, back in the case of the Hydrogen atom, the negative energy states only fill half of $L^2(\mathbf{R}^3)$, the other half being a direct integral of nonbound states. Which brings us to scattering (and inverse scattering [15]) theory [12] and poles of the scattering matrix. In physical practice, elementary particles are often associated with resonances in a scattering experiment, and can sometimes be thought of as the (Regge) poles of the scattering matrix. These poles empirically lie on a logarithmic curve, a fact which still has to be understood mathematically (not to mention the number-theoretic meaning of these poles for the automorphic wave equation [13]). Finally, one might discuss states of operator algebras as a vehicle for statistical mechanics, in turn a model of quantum field theory.

Not the least bit discouraged, everybody signs up for the next year of manifolds: Starting with sheaf cohomology and various realizations such as de Rham cohomology and characteristic classes of fiber bundles, one could include the Riemann-Roch theorem to make way for some surprising applications to the unrelativisitic Hydrogen atom [19]. Morse theory has most recently been used [3] in studying the homology of the space of instanton solutions to the Euclidean Yang-Mills equations, and even classical applications to certain density functions in crystals are still interesting [25]. Topological dynamics and phase-plane analysis with bifurcations are of course not just of relevance in physics but also in much mathematical biology and chemistry. Somewhat in the same vein one might conclude with singularities of differential mappings plus unfoldings and various indices of singularities.

Any treatment of physics must necessarily be based on examples close to nature; little test particles make their appearance, small quantities sometimes vanish altogether (sometimes even big ones)–but foremost one must spend some time on the philosophy of the relation between the coherence of the physical system on one side and that of the mathematics describing it on the other. Somewhat crudely, one could say that it took Einstein longer to find the equations of gravity than it took Hilbert, who merely from a natural axiom derived its consequences. But then Einstein's understanding and appreciation was that much deeper than Hilbert's. General relativity [17] could be an example of action principles and also illustrate the difficulties in giving a system both a Lagrangian formulation and a canonical formulation in terms of a Hamiltonian on a symplectic manifold. Gauge field theories (see [21] for a good survey of this and other geometrical aspects of particle physics) inherit some of their interpretation from the gauge invariance of Maxwell's equations. Therefore, it is important to get a clear picture of (classical) relativistic particle dynamics in the space-time M of general relativity. One formulation considers the cotangent bundle T^*M of M with canonical projection π and 2-form ω. Let H be the Hamiltonian defined by the Lorentz square of a tangent vector (transferred to T^*M via the same Lorentz metric on M). Compute the Hamiltonian vector field X of H relative to the form $\omega + \pi^*F$, where F is the 2-form on M representing the electromagnetic field. Then the flow of X gives the movement of the particle. *WKB*-approximations to solutions of Schrödinger's equation and geometric quantization [7] would bring us back full circle to recent developments in representations of semisimple Lie groups [10] (*WKB* becomes exact and not just an approximation for the character of a discrete series representation constructed from a certain Lagrangian in the cotangent bundle of the group–a recent unpublished result by B. Kostant, see also [16], [24]).

Wishful thinking aside, how does the volume by Choquet, de Witt and Dillard prepare the student (this is not a research book) for the world of analysis, manifolds and physics? A good part of the book deals with basic differential geometry, somewhat in the spirit of [6], giving a lucid exposition under the steady hand of Madame Choquet-Bruhat. Care is taken in balancing concrete understanding in coordinate charts against avoiding too may confusing tensor indices. This section includes exterior differential systems, their characteristic manifolds, complete integrability and integration on manifolds with a good introduction to homology and cohomology. Pseudo-Riemannian manifolds and connections are illuminated by the canonical differential operators on differential forms and covariant differentiation respectively, followed by a brief discussion of geodesics on Riemannian manifolds.

The next chapter gives the definitions and properties of spaces of distributions, convolutions and Fourier transforms and applications to Sobolev spaces and partial differential equations. Finally there is a presentation of the prerequisites for the study of nonlinear partial differential equations, namely infinite dimensional manifolds (with cylindrical measures in the linear case) and Leray-Schauder theory using degree of mappings to establish existence of solutions.

It is impossible in a few words to do justice to this voluminous work which contains much information (even the introductory chapter listing background from analysis)–indeed I think it will be valuable also to physicists wanting to look up facts in analysis on manifolds. Many of the longer proofs are omitted, but there is still much to learn from the remaining proofs. The format of the book is carefully worked out with index and important definitions marked in the margin of each page, and examples are well placed in the text.

All applications to physics are relegated to problems at the end of each chapter with answers immediately following. Sometimes in these exercises one feels that things are piled a little too high with many definitions, interpretations and solutions obscuring just where the physics fits into the mathematics. Perhaps a few more words on how the Hamilton-Jacobi equation is related to the calculus of variations or to Hamilton's equation would be well placed in problem 4, p. 260; or in problem 2, p. 104 we are told (and likewise several other places) that a formula for the Van Vleck matrix plays an important role in many problems of physics–leaving us wanting to know what role and what physics. Still, for the energetic beginning student of physical mathematics there are enough references to lead him on, and certainly many hints in the problems aimed at spinors, Jacobi fields, Euler-Lagrange equations, soap-bubbles, Maxwell's equations, catastrophes, Hamiltonians, shock waves, the Schwarzschild solution, wave equations, the diffusion equation, the symplectic structure of the Klein-Gordon equation, Wiener integrals and more.

Obviously many topics are missing in the book, but I especially would have liked to see some symplectic geometry, Yang-Mills fields, groups of symmetries, fluid dynamics and examples like mechanics of deformable bodies (say formulating Hooke's law) and a little kinematics of fields and particles on curved manifolds.

As science is becoming more and more compartmentalized, it is encouraging with enterprises that want to cross the waters between mathematics and physics and let the indigenous populations see through the mutual mist. Far from raising any sunken Atlantis, I still recommend this book and its authors for visualizing the voyage and for setting up a firm outpost on mainland mathematics.

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Banach modules and functors on categories of Banach spaces, by Johann Cigler, Viktor Losert, and Peter Michor, Lecture Notes in Pure and Applied Mathematics, Volume 46, Marcel Dekker, New York, 1979, xviii + 282 pp., $29.50.

As the authors state in their preface, this is a book about "general nonsense", a term indicating the uneasy attitude many of us have towards the material. This term cannot be other than perjorative–why should a valid and necessary part of an argument get such scant respect? Many of us lose patience with a tower of increasingly complicated general propositions with a liberal scattering of words like natural and contravariant with perhaps a diagram which when chased enough merely states the obvious–why can't we stick to something interesting like operator theory where there are real theorems? And yet there must be another side of the coin or the subject would not attract the attention of enough competent mathematicians to survive–what can it be? One ingredient in our reaction is the reluctance to take a new point of view, learn some new words and a new way of looking at things. Former generations reacted similarly to modern analysis and abstract algebra. However some notions really do need this generalized framework, for example the concept of a tensor norm. Often this is defined as a norm on a product $X \otimes Y$ of Banach spaces but the way the term is used is more in keeping with thinking of it as a description of a norm on each possible $X \otimes Y$ with various relationships between the norms so described–if you accept this