11. M. Olinick, *An introduction to mathematical models in the social and life sciences*, Addison-Wesley, Reading, Mass., 1978.

12. W. H. Riker and S. J. Brams, *The Paradox of vote trading*, American Political Science Review 67 (1973), 1235–1247.

13. F. Roberts, *Discrete mathematical models: with applications to social biological, and environmental problems*, Prentice-Hall, Englewood Cliffs, N. J., 1976.

14. T. L. Saaty and F. J. Wegl, *The spirit and the uses of the mathematical sciences*, McGraw-Hill, New York, 1969.

15. E. M. Uslaner and J. R. Davis, *The paradox of vote trading: effects of decision rules and voting strategies on externalities*, American Political Science Review 69 (1975), 929–942.

DANIEL P. MAKI

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, July 1980

0002-9904/80/0000-0317/$01.50

Mathematics of finite-dimensional control systems. Theory and design, by David L. Russell, Lecture Notes in Pure and Applied Mathematics, Vol. 43, Marcel Dekker, New York and Basel, 1979, viii + 553 pp., $45.00.

Control theory was brought into existence during the second half of the eighteenth century by the development of complex machinery such as the steam engine. Since that time until about 1900 it was primarily concerned with elimination of undesirable traits (chiefly instability) by means of feedback devices, the Watt governor being a notable example; design was mainly the result of intuition and empirical insights. The beginning of the theory can be traced to J. C. Maxwell's celebrated paper on governors [1]. Progress was slow during the nineteenth century but became faster after 1900 due to the development of power transmission, communications and complex processing plants and some mathematical techniques (such as the Routh-Hurwitz stability criteria) began to be systematically used. Growth was enormous during and after the second World War and many other mathematical tools like Laplace transforms and probability theory found applications. In the late fifties and early sixties, starting with the work of Bellman, Glicksberg and Gross [2], Bellman [3], Pontryagin, Boltyanskiĭ, Gamkrelidze and Mischenko [4], Kalman [5], Kalman and Bucy [6] and others, control theory began to be accepted as a respectable mathematical discipline. It also started to absorb relatively sophisticated "modern" mathematics into its language (for instance measure theory, elementary functional analysis, abstract algebra and Liapunov stability theory) and brought to the forefront the idea of *quality of control*: if the control engineer was content in the past, say, with rendering stable the operation of a machine by means of a feedback device his modern counterpart would try to achieve the same effect in a suitably optimal way (for instance, minimizing the stabilization time, the cost of the control device, the strain on the machinery, etc.). Finally, concepts like controllability, observability and stabilization by feedback, until then living in a latent state in the literature were given precise formulations.

Although many of the initial contributions to the mathematical theory of control were firmly rooted in reality (for instance, the influence of [3] and [6] in modern technology was and is enormous) control theory tended to develop along two parallel lines since the early sixties. The first is practiced by mathematicians, relies on mathematics that the average engineer considers esoteric and (at least in some cases) has tended to produce more general theory than solutions of particular problems. On the positive side, it has brought to light connections (such as that between optimal control theory and classical variational calculus), it has clarified many concepts and eliminated duplication. The second line inherits the classical approach and stresses the need for solving real problems, some times neglecting the search for general principles and ignoring the difference between an heuristic argument and a proof. It must be recognized of course that here, as in almost all areas of applied science, mathematics lags behind the applications. Although the overlap of these two fields of research is considerable, a great part of the literature can be easily placed in one (and only one) of them.

This is not the case with the book under review. In spite of its title, I believe it will be of interest both to mathematicians and engineers. The author is a mathematician and a practitioner of control theory and both qualifications are obvious throughout the book. Proofs are complete and correct (as well as elementary and self-contained when possible) but concrete problems are never out of sight and the actual design and implementation of control devices, including computation of all parameters involved is stressed at all times. The mathematical demands on the reader do not go much beyond linear algebra and multivariable calculus; all that is needed from the theory of differential equations can be found in Chapter I and the Lebesgue integral can be dispensed with if one is willing to accept on faith a few existence statements. Some acquaintance with probability may be necessary in Chapter VI but all nonelementary facts needed (such as those from the theory of stochastic differential equations) are carefully introduced and explained.

The book considers exclusively control systems described by systems of ordinary differential equations. For the most part these systems are deterministic, but stochastic systems (in particular, deterministic systems where the state is corrupted by white noise) are studied at length in Chapter VI. The equations are mostly linear, but there are several exceptions; controllability of a nonlinear system near an equilibrium point is found in Chapter II and some results on global controllability are given in Chapter IV. The treatment of optimal control problems is chiefly restricted to linear systems with quadratic performance criteria, both in finite and infinite time intervals for deterministic and stochastic systems, although the dynamic programming method for a general optimal control problem is briefly expounded in Chapter IV. A great deal of attention is given to computational methods, especially in Chapter V where numerical solution of the quadratic matrix equations arising from linear-quadratic problems is discussed at length.

An especially attractive feature of this book is the nice balance between the classical and the abstract. For instance, after controllability and observability are given a suitably elementary treatment an abstract version of these concepts (due to Doleckiĭ and Russell) is introduced; the reader who knows (or is willing to learn) a few elementary facts on operators in Hilbert space will be able to understand the subject in a deeper and more transparent fashion. Another is the awareness, obvious in many places but especially in p. 93 of the divergent points of view of the engineer and the mathematician about what is meant by "optimal". Finally, mention should be made of the excellent collection of exercises, some of them challenging numerical projects involving access to a high speed computer.

There are not many references in the literature where one can learn of real control problems without undue strain on credibility. This book is one of them.

REFERENCES

1. J. C. Maxwell, *On governors*, Proc. Royal Soc. **16** (1868), 270–283.

2. R. Bellman, I. Glicksberg and O. Gross, *On the "bang-bang" control problem*, Quart. Appl. Math. **14** (1956), 11–18.

3. R. Bellman, *Dynamic programming*, Princeton Univ. Press, Princeton, N. J., 1957.

4. L. S. Pontryagin, V. G. Boltyanskiĭ, R. V. Gamkrelidze and E. F. Mischenko, *The mathematical theory of optimal processes*, Gostekhizdat, Moscow, 1961. English translation: Wiley, New York, 1962.

5. R. E. Kalman, *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana **12** (1960), 102–119.

6. R. E. Kalman and R. S. Bucy, *New results in linear prediction and filtering theory*, J. Basic Eng. (Trans. ASME, Ser. D) **83** (1961), 95–107.

H. O. FATTORINI

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, July 1980

0002 9904/80/0000-0318/$01.75

Basic set theory, by Azriel Levy, Springer-Verlag, Berlin, Heidelberg, New York, 1979, xiv + 391 pp., $24.90.

Perhaps the greatest obstacle in teaching elementary set theory is the mathematical logic needed to formalize the axioms. There is nothing inherently difficult about the basic material–the theory of ordinal and cardinal numbers, and the axiom of choice–which every mathematician is expected to know. And most of the axioms of ZF, the system of Zermelo and Fraenkel used most frequently nowadays, can be stated easily and understandably in English. The exception is the axiom scheme of replacement, which when formalized looks like this

$$\forall x_1 \cdots \forall x_n (\forall x\, \forall y\, \forall z(\varphi(x, y, x_1, \ldots, x_n) \wedge \varphi(x, z, x_1, \ldots, x_n) \rightarrow y = z) \\ \rightarrow \forall a\, \exists b\, \forall w(w \in b \leftrightarrow \exists x(x \in a \wedge \varphi(x, w, x_1, \ldots, x_n)))).$$

Here φ stands for a formula of the first-order language of set theory; each such φ yields a new instance of the axiom scheme, so there are infinitely many axioms.

Now the idea behind the replacement scheme is quite simple: any correspondence carries sets to sets. The problem is how the "correspondence" is to be specified. The solution, of course, is that the correspondence must be definable from parameters in a way which could be formalized in first-order logic. Unfortunately, many students do not find this completely clear.

Nor is this the only such problem. To take another example, each instance of the principle of definition by transfinite recursion is usually quite clear, yet the formalization of the principle itself (as a theorem scheme) is often confusing.