manifolds. Another type of generalization, partly due to Kawai, Kashiwara, Sato and partly due to Hörmander, involves the notion of the "wave front set" of a distribution as a kind of substitute for the holonomic systems mentioned above. For instance, Hörmander shows that if $f: X \to Y$ is a smooth map and φ is a generalized function on Y then $f^*\varphi$ can be defined in a legitimate way if f is transversal to the wave front set of φ. More generally he shows that both the pull-back and push-forward operations behave well with respect to maps which are well-situated (transversal) with respect to the wave-front sets of the distributions to which these operations are applied. Moreover, the "Fourier integral distributions" which have come to play such a central role in the theory of hyperbolic differential equations lately, turn out to be *exactly those distributions which can be generated from the distributions* $(x + 0i)^\lambda$ *on the real line by pull-back and push-forward operations satisfying these transversality conditions.*

There are a number of topics in volumes 1 and 5 of *Generalized functions* which seem capable of further exploitation. For instance in spite of the impetus given to the field of integral geometry by the work of Gelfand-Graev-Vikenkin on the Plancherel formula for $SL(2, \mathbf{C})$ we still know embarrassingly little about these questions. In volume 5, Gelfand et al give necessary and sufficient conditions for a line complex in CP^3 to be "admissible", i.e. to have the property that the integrals of a function on CP^3 over the lines of the complex determine it unequivically. Later Gelfand and Graev extended this result to CP^n; however we still do not know much about the admissibility of complexes of planes, three-folds etc.

Another topic which deserves further investigation is the question of what class of distributions one gets if, starting with the elementary distributions, $z^\mu \bar{z}^\nu$, on the complex line, one tries to generate new distributions by successive "pull-backs" and "push-forwards". (For the beginnings of a theory, see appendix B of volume 1.)

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Schwartz spaces, nuclear spaces and tensor products, by Yau-Chuen Wong, Lecture Notes in Math., vol 726, Springer-Verlag, Berlin-Heidelberg-New York, 1979, viii + 418 pp., $19.50.

During a conversation in 1966, G. Köthe said to me, "The best book about the general theory of nuclear spaces and tensor products in existence today is the book of A. Pietsch [16]". Perhaps the kindest thing to be said about the present book is that Köthe's statement is still true. In fact, although he tries to hide it with new names such as "prenuclear norm", most of Wong's book could have been written at the time of that conversation. He doesn't write very much about what has happened in the ensuing 14 years.

It is common knowledge that the origin of the theory of nuclear spaces (and tensor products, too—there is little to be said about Schwartz Spaces these days) lies in the thesis of A. Grothendieck [10]. I'll indicate some definitions later, but first a little history. Initially, there was very little progress. Grothendieck had given a lot of attention to showing that many really different statements were, in fact, equivalent definitions of nuclearity. It is an unfortunate aspect of the history of this subject that so many authors do not realize that an important feature of Grothendieck's equivalences were that they were not obvious. These days we get a myriad of "new" characterizations (as in the present book) of nuclearity which are no more than juggling definitions and/or minor variations on one of Grothendieck's themes.

The first post-Grothendieck result of consequence appeared in 1960 in a paper of A. S. Dynin and B. S. Mityagin [8] who showed that every basis in a nuclear Fréchet space is *absolute* in the sense that the series expansion of each element is absolutely convergent with respect to any continuous seminorm on the space. This was quickly followed by the landmark paper of Mityagin [13] who developed the connection between bases and certain linear topological invariants.

At this point there was a pause in the development of the theory and perhaps I, also, should pause to give a few details about the subject and Wong's book.

Any locally convex space has a unique completion (because it is a uniform space) and this completion is a projective limit of Banach spaces. The most useful classifications have come from making assumptions about the linking maps in the projective limit. For example if you assume that the maps are compact you obtain the class of Schwartz spaces. If you assume that the maps are absolutely summing or, equivalently, that they are Hilbert-Schmidt (so you have also assumed that the Banach spaces are Hilbert) you obtain the class of nuclear spaces. And so on. Now I have told you what is in the first half of the book.

Tensor products (from the algebraic point of view) can be interpreted as spaces of linear maps or as spaces of bilinear forms. Thus the various tensor product topologies can be interpreted in several different ways. In the presence of nuclearity all of the tensor product topologies coincide and this statement has several different formulations corresponding to the different interpretations of the tensor product. This describes another quarter of the book.

Add some familiar consequences of the Dynin-Mityagin theorem, which connects nuclear spaces and sequence spaces, a proof of the Kōmura-Kōmura embedding theorem along with a totally incomprehensible chapter on order properties and you have the whole book.

To sum up, I can think of no situation in which I would suggest that someone read this book and I question the decision to publish it.

It may be that it is best to leave the matter at this point. But I am concerned that the appearance of this book may suggest that research in the theory of nuclear spaces over the last decade and a half consists entirely of reworking the foundations of the theory, juggling highly technical terminology to obtain "new" results, and calling theorem that which is proved by straightforward verification. Certainly, I am sorry to admit, there is a large amount of literature to support such a suggestion.

However, exactly the opposite is true! Ignoring the junk (which I am sure abounds in every field of mathematics), it is possible to discern a body of work which is not only solid but even exciting. The number of papers is not large and the number of authors is small. But there has been some good taste shown, several virtuoso technical performances, and even a touch of artistic creativity. Let me tell you a little about it.

It is easy to summarize the major achievements in the theory of nuclear spaces over the last 14 years. Three problems (all stated by Grothendieck) have been solved and a fourth (stated by M. M. Dragilev) has been solved in several important cases. In addition, some new lines of investigation have been opened up, new techniques have been invented and new problems are being generated.

Now for some details.

I'll begin with the universal space problem. In view of the well-known fact that every separable Banach space is a subspace of the Banach space $C[0, 1]$ of continuous functions on the unit interval, it is natural to try to find a function space that does the same job for nuclear Fréchet spaces (that is, nuclear locally convex spaces which are complete and metrizable). Grothendieck [10] proposed $C^{\infty}(\mathbf{R})$, the infinitely differentiable functions on the real line with the topology of uniform convergence of each derivative on each compact set. It is easy to see [16] that $C^{\infty}(\mathbf{R})$ is isomorphic to a certain sequence space $(s)^{\mathbf{N}}$ and in 1966 T. Kōmura and Y. Kōmura [12] showed that, yes, every nuclear Fréchet space is a subspace of $(s)^{\mathbf{N}}$.

This fact has proved useful. For example it is an essential ingredient in Vogt's method of studying subspaces of a particular space [17].

Next, consider the basis problem. A sequence (x_m) in a locally convex space E is a *basis* for E if $\forall x \in E\ \exists$ a unique sequence (t_m) of scalars $\ni x = \sum_m t_m x_m$ (convergence in the topology of E). The question is whether every E in a given class has a basis. For the class of separable Banach spaces this question and its (negative) solution by P. Enflo [9] has been one of the central issues in Functional Analysis for about 40 years. Grothendieck asked this question for nuclear Fréchet spaces in 1955 [10]. It was answered (again in the negative) by Mityagin and N. M. Zobin in 1974 [15].

The study of bases continued beyond the solution of the main problem. There was an investigation of whether the counterexamples could appear as subspaces or quotient spaces of "nice" spaces. The unfortunate ubiquity of the pathology was established by C. Bessaga, this reviewer and Mityagin in [1], [5] and [7].

As most people know, the basis problem in Banach spaces is closely connected with the approximation problem–that is, does there always exist in a Banach space E a net of operators with finite-dimensional range converging to the identity uniformly on compact sets. This is the question that Enflo [9] answered in the negative. However, it is a standard fact that in any nuclear Fréchet space such a net does exist. So Grothendieck posed a harder problem (interesting also for Banach spaces). Can you always do it so that the net of operators is bounded? (By the uniform boundedness principle you can define a bounded set of operators just about any way you like.) Again the answer is no and this was established in 1978 [6].

Present research on this bounded approximation problem is about the ubiquity issue. There are also some intriguing variations of the definition that are being investigated.

The last problem, and the one still only partially solved is the quasi-equivalence problem of M. M. Dragilev [4]. If (x_m) and (y_m) are two bases in a nuclear Fréchet space E it is often possible to find a sequence (t_m) of positive scalars, a permutation π of the indices and an isomorphism $T: E \to E \ni Tx_m = t_m y_{\pi(m)}$. In this case we say that (x_m) and (y_m) are *quasi-equivalent*. The question is whether all bases in a nuclear Fréchet space E are quasi-equivalent. Essentially this would mean that the basis in E (if it exists) is "unique". There are now many classes of nuclear Fréchet spaces for which this question has a positive answer. These are described by Dragilev [4], Mityagin [14], V. P. Zahariuta [18], L. Crone and W. B. Robinson [3].

Of course there are many problems and results which I have not mentioned–concerning spaces of analytic functions, complemented subspaces, etc. But perhaps the above will suffice to give a flavor of the questions and answers in the subject–if not the methods.

Finally I would like to make brief mention of some new areas in which research has been made and where I expect to see additional progress in the future.

G. M. Henkin and Mityagin [11] have considered linearized versions of some classical problems of the theory of functions of several complex variables. For example if M is a closed submanifold of G then $H(M)$, $H(G)$–the spaces of holomorphic functions on M, G–are nuclear Fréchet spaces and the Oka-Cartan theorem says that the restriction operator R: $H(G) \to H(M)$ is surjective–that is, $\forall f \in H(M)\ \exists \tilde{f} \in H(G) \ni \tilde{f}|_M = f$. But can this extension be made so that $\tilde{f}$ depends linearly and continuously on f? This is equivalent to requiring that the kernel of R splits in $H(G)$. The answer seems to depend on M, G.

Using ideas of Mityagin [11], Zahariuta [19] has introduced new linear topological invariants for nuclear spaces and has used them to study isomorphism classes.

In several papers there is developing a surprisingly complete analysis of all subspaces and quotient spaces of a fixed nuclear Fréchet space. This can be done, for example, with stable power series spaces including some spaces of analytic functions. Actually, the development has taken place in two directions. In one, certain simple conditions on the neighborhoods and the asymptotic behavior of Kolmogorov diameters lead to a complete characterization. This has mostly been done by D. Vogt and M.-J. Wagner. The basic ideas are described by Vogt in [17]. In the other, more detailed information is obtained but only subspaces with bases are characterized. This approach is described in [6].

Quite recently there has been appearing a growing literature about spaces of holomorphic functions defined on the dual of a nuclear Fréchet space. This is a branch of infinite-dimensional holomorphy and it yields (apparently) new examples of nuclear Fréchet spaces. Perhaps the most interesting results are in a paper by Börgens, Meise and Vogt [2].

In conclusion it occurs to me that it was perhaps unfair to use most of my space writing about the book Wong should have written, rather than the one he did write. But, as I've indicated, I don't think much of it, so I've tried to express some of the excitement I feel about research in this field. The real danger in writing a book is that you can turn people off and conceal the enthusiasm that workers in the subject have. I hope I've counteracted that a little.

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ED DUBINSKY

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Mathematical modelling techniques, by R. Aris, Research Notes in Mathematics, no. 24, Pitman, London-San Francisco-Melbourne, 1979, x + 191 pp., $15.00.

Forming and studying mathematical models and talking about forming and studying mathematical models are both quite fashionable right now. While the activity of model building (the forming and studying) has no doubt been