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Generalized functions: Volume 1, *Properties and operations*, xviii + 423 pp.; Volume 2, *Spaces of fundamental and generalized functions*, x + 261 pp.; Volume 3, *Theory of differential equations*, x + 222 pp.; Volume 4, *Applications of harmonic analysis*, xiv + 384 pp.; Volume 5, *Integral geometry and representation theory*, xvii + 449 pp.; by I. M. Gel'fand and G. E. Shilov, Academic Press, New York and London, 1977.

At the beginning of the 1950's the theory of generalized functions was in somewhat the same state that nonstandard analysis is in today. Mathematicians were by no means of one mind as regards the benefits of the theory. Critics felt that it was an overblown way of describing a modest but useful scheme for making computations in one area of harmonic analysis: the Heaviside calculus. Even those who were enthusiastic about the theory regarded distributions as shadowy entities like quarks or mirons. It was felt that to understand the theory one had first to become familiar with a formidable array of topics in abstract analysis: barrelled topological vector spaces, Montel spaces and so on. Graduate students were discouraged from going into distribution theory and advised to do Schauder-Leray estimates instead.

By the end of the 50s this situation had completely changed. Generalized functions had come to be viewed as an indispensible tool in almost every area of analysis. The reasons for this were not hard to account for. The novelty of the theory wore off and people gradually got used to thinking of distributions as house-and-garden variety objects. It turned out that the function-theoretic underpinnings of the theory could be reduced to standard facts about Sobolev spaces, so one did not need to know about espaces tonnellés. In fact to learn enough of the theory to be able to work with distributions, albeit nonrigorously, one could get by with a few elementary facts about the Fourier transform. This meant that distributions could be made a regular part of the graduate curriculum. Finally, two extremely important mathematical developments, both occurring in the middle 50s, turned out to depend on the theory of distributions in an absolutely essential way. One of these was in the area of linear partial differential equations. In 1955, Ehrenpreis, Hörmander, and Malgrange proved independently that every constant coefficient partial differential equation admits a fundamental solution. The solution is produced by making sense of $p(\xi)^{-1}$ as a generalized function when p is a polynomial function on $\mathbf{R}^n$. There are several ways of doing this, but all require Schwartz's theory of distributions.

The other development was in the area of group representations. In his thesis Bruhat was able to settle a number of fundamental questions concerning the irreducibility and unitarizability of induced representations of Lie groups by reducing them to technical questions about the kernels of intertwining operators. The mathematical tool which made this reduction possible was one of the key theorems in distribution theory, the Schwartz kernel theorem.

The five volumes of *Generalized functions* were mostly written while these developments were taking place and, in fact, in no small way contributed to these developments. On rereading these volumes after a hiatus of several years, the reviewer was struck by the amount of innovative mathematics they contain: One can find many explicit formulas for the Fourier transforms of distributions in the engineering literature and more such in Schwartz and in Marcel Riesz's Acta paper, but the list of such results in volume one is certainly the most extensive such list ever compiled. (Among other things, this list provides explicit formulas for the fundamental solutions of many interesting partial differential equations.) I mentioned above the many technical improvements in the theory of distributions that occurred during the fifties. A number of these can be found in volumes 2 and 4, e.g. the notion of "rigged Hilbert spaces". The tie-ins between spectral theory and distribution theory, a topic which has been of major importance in recent years, really started with the notion of "generalized eigenfunctions" in volume 3. In volume 5 one finds the Bruhat theory which I commented on above and the complementary "Plancherel" theory worked out in complete detail for the first time for the groups $SL(2, \mathbf{C})$ and $SL(2, \mathbf{R})$. One also sees spelled out the role of integral geometry in this theory, a topic which I will have more to say about below. I do not want to attempt here an item-by-item inventory of the contents of these volumes; but I hope the examples above will suffice to convey some sense of the amount of original material they contain.

In this review I would like to comment mainly on the material in volumes 1 and 5. What impresses one about these volumes is their remarkable cohesiveness of tone, in spite of the diversity of the subjects that they deal with. I think this is attributable to the fact that Gelfand and his co-authors are committed to thinking about distributions in a rather novel way, namely not just as mathematical objects to be studied "in vitro" but as objects to be studied "in vivo", i.e. objects in the sense of category theory subject to the dynamics of functional operations. To explain what I mean by this, let me make a few trivial remarks about the functorial properties of distributions. C^∞ functions per se are "contravariant" objects. If X and Y are manifolds and f: $X \to Y$ a smooth, proper map then C_0^∞ functions on Y "pull back" via f to C_0^∞ functions on X. Generalized densities, being the dual objects to C_0^∞ functions have the opposite variance; they "push forward". Moreover, in certain instances this "push forward" operation carries the subspace of C_0^∞ densities into itself. This happens for example if f is a proper submersion. In this case, since generalized functions are the dual objects to C_0^∞ densities, one gets a corresponding "pull-back" operation on generalized functions. To summarize, let f: $X \to Y$ be a smooth proper mapping. Then

(A) generalized densities "push forward" under f and

(B) generalized functions "pull back" under f if f is a submersion.

It is easy to see that many of the basic operations on functions and distributions are amalgams of these two basic operations. To take a simple example consider the ordinary product, $\varphi\psi$, of functions φ and ψ on X. This product is the "pull-back" via the diagonal map $X \to X \times X$ of the function $\varphi \times \psi$ on $X \times X$. To take another example, let K: $C^\infty X \to C^\infty Y$ and L: $C^\infty Y \to C^\infty Z$ be integral operators with kernels $\varphi(x, y)$ and $\psi(y, z)$. Then the kernel of the operator LK is the pull-back of $\varphi \times \psi$ via $X \times Y \times Z \to X \times Y \times Y \times Z$ followed by the push-forward via $X \times Y \times Z \to X \times Z$.

One of the *idées clefs* of volumes 1 and 5 is that most of the generalized functions that come up in concrete problems are obtainable by starting with a small supply of simple distributions on the real line, e.g. x_+^λ, $\theta(x)$, $\delta(x)$, $(x + 0i)^\lambda$ etc. and applying to them sequences of functorial operations. This idea is not new with Gelfand et al.; it was certainly familiar to Hadamard and Riesz; however, what is remarkable about *Generalized functions* is the pertinacity with which this idea is developed, the number of examples elicited in illustration of it. Another *ideé clef* is that the central problem of distribution theory is to legitimatize the operations (A) and (B) when f happens not to be proper or happens not to be a submersion. For instance consider the problem of "pulling back" the distributions x_+^λ, $\delta(x)$, etc. on the real line with respect to the map p; $\mathbf{R}^n \to \mathbf{R}$. If p is a polynomial and $p \geqslant 0$, to make sense of p^{-1} as a distribution (the problem of Ehrenpreis-Hörmander-Malgrange mentioned earlier) amounts to defining the pull-back of x_+^λ for $\lambda = -1$. If $\operatorname{Re} \lambda > 0$, x_+^λ is a continuous function; so the usual pull-back is well-defined. One way to make sense of the pull-back of x_+^λ for $\lambda = -1$ is by analytically continuing λ from $\operatorname{Re} \lambda \geqslant 0$ to $\lambda = -1$. In volume 1, it is shown that this is legitimate for large classes of p's.

The problems in integral geometry discussed in Chapter 5 are also problems of this nature. The most general sorts of integral transforms are transforms of the form, $g_* f^*$, where f: $Z \to X$ and g: $Z \to Y$ are submersions. Obtaining an inversion formula for such a transform involves determining the fate of the delta function, δ_{x_0}, for arbitrary $x_0 \in X$ under successive applications of f^*, g_*, g^* and f_*. (This of course involves legitimatizing the application of those operations on distributions.) As Gelfand et al. show in volume 5 this problem, which seems to involve four functorial operations can very often be reduced to a problem involving just one functorial operation. For instance let p: $\mathbf{R}^n \to \mathbf{R}$ be a polynomial function with an isolated singularity at the origin, and consider the following problem in integral geometry: determine the value at 0 of an arbitrary smooth, compactly supported function φ from the integrals of φ over hypersurfaces $p = c$, $c > 0$. It is clear that these integrals determine $\langle p_+^\lambda, \varphi \rangle$ for all λ. Under some mild assumptions on the nature of the singularity of p at 0, the distribution p_+^λ is known to have residues for certain rational values of λ which are either δ-functions or derivatives of δ-functions; so in these cases we *can* determine $\varphi(0)$ from the integral data. An example for which this happens is when $n = 2m$ and $p = p(x, y) = x_1 y_1 + \cdots + x_n y_n$. Applying the argument above to this example one gets a very elegant proof of the classical Radon inversion formula.

As an explicit illustration of these ideas let me describe the derivation given in volume 5 of the Plancherel theory for $SL(2, \mathbf{C})$. Let $G = SL(2, \mathbf{C})$ and let

$$N = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, c \in C \right\}$$

be the maximal unipotent subgroup of G. By a *horocycle* in G one means any subset of G of the form aNb. The space of all horocycles, which we will denote by $G^\#$; is a three dimensional complex manifold on which $G \times G$

acts in a transitive fashion. Moreover, a typical left G-orbit in $G^{\#}$ is of the form G/N. It is not hard, therefore, to obtain a "Plancherel theory" for $L^2(G^{\#})$ (i.e. decompose $L^2(G^{\#})$ into a direct integral of subspaces invariant under the left G-action) by means of the theory of the principal series: the Plancherel formula for $L^2(G/N)$. The idea is to relate the Plancherel theorem on $G^{\#}$ to the Plancherel theorem on G by means of the obvious intertwining operator "integration over horocycles". Thus we have more or less reduced our problem to the following problem in integral geometry: determine a function on G from its integrals over horocycles. This in turn can be reduced to an even more elementary problem. The group $SL(2, \mathbf{C})$ is the submanifold of $\mathbf{C}^4$ defined by the quadratic equation

$$Q(z) = z_1 z_3 - z_2 z_4 - 1 = 0.$$

It is easy to see that the horocycles are precisely the complex lines in $\mathbf{C}^4$ which lie on this hypersurface. Given a function φ on the hypersurface. $Q = 0$, we can associate with it the distribution $\varphi Q^* \delta_0$, δ_0 being the delta function at the origin on the complex line. By the (complex) Radon inversion formula, whose derivative we sketched above, $\varphi Q^* \delta_0$ is determinable by its integrals over the hyperplanes, H, in $\mathbf{C}^4$. Therefore φ is determined by its integrals over these hyperplanes intersected with $Q = 0$. However, it is easy to see that the intersection of H with $Q = 0$ is ruled by lines; so these integrals are in turn determined by the integrals over horocycles. Q.E.D.

To reiterate, I feel the idées clefs in volumes 1 and 5 are the observations that:

I. Most of the interesting generalized functions in analysis are obtained from simple distributions on the real line like x_+^λ and $\delta(x)$ by means of the functorial operations "pull-back" and "push-forward".

II. The basic problem in the theory of generalized functions is to legitimatize these functorial operations for maps which fail to satisfy conditions (A) and (B).

It is only in recent years that the significance of these two ideas has come to be duly appreciated, and I would like to conclude this review by describing some of the recent history of distribution theory in order to show just how influential these ideas have been. To begin with, Bernstein showed in 1973 that for polynomial mappings between Euclidean spaces the "pull-back" and "push-forward" operations could always be legitimatized for distributions generated by such operations from the elementary distributions on the real line. The proof of this fact involved a remarkable observation: namely, distributions of this sort always satisfy large numbers of differential equations with polynomial coefficients (so-called "holonomic systems"). These equations not only enable one to define "pull-back" and push-forward" operations by the kinds of analytic continuation arguments described above but also enable one to establish some surprising analytic properties for these distributions: real analyticity off semialgebraic sets, existence of analytic continuations to the complex domain, etc.

Bernstein's results apply to polynomial mappings between Euclidean spaces. Kawai, Kashiwara, Sato and Bjork showed that much of the Bernstein theory could be generalized to real analytic mappings between real analytic manifolds. Another type of generalization, partly due to Kawai, Kashiwara, Sato and partly due to Hörmander, involves the notion of the "wave front set" of a distribution as a kind of substitute for the holonomic systems mentioned above. For instance, Hörmander shows that if $f\colon X \to Y$ is a smooth map and φ is a generalized function on Y then $f^*\varphi$ can be defined in a legitimate way if f is transversal to the wave front set of φ. More generally he shows that both the pull-back and push-forward operations behave well with respect to maps which are well-situated (transversal) with respect to the wave-front sets of the distributions to which these operations are applied. Moreover, the "Fourier integral distributions" which have come to play such a central role in the theory of hyperbolic differential equations lately, turn out to be *exactly those distributions which can be generated from the distributions* $(x + 0i)^\lambda$ *on the real line by pull-back and push-forward operations satisfying these transversality conditions.*

There are a number of topics in volumes 1 and 5 of *Generalized functions* which seem capable of further exploitation. For instance in spite of the impetus given to the field of integral geometry by the work of Gelfand-Graev-Vikenkin on the Plancherel formula for $SL(2, \mathbf{C})$ we still know embarrassingly little about these questions. In volume 5, Gelfand et al give necessary and sufficient conditions for a line complex in CP^3 to be "admissible", i.e. to have the property that the integrals of a function on CP^3 over the lines of the complex determine it unequivically. Later Gelfand and Graev extended this result to CP^n; however we still do not know much about the admissibility of complexes of planes, three-folds etc.

Another topic which deserves further investigation is the question of what class of distributions one gets if, starting with the elementary distributions, $z^\mu \bar{z}^\nu$, on the complex line, one tries to generate new distributions by successive "pull-backs" and "push-forwards". (For the beginnings of a theory, see appendix B of volume 1.)

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Schwartz spaces, nuclear spaces and tensor products, by Yau-Chuen Wong, Lecture Notes in Math., vol 726, Springer-Verlag, Berlin-Heidelberg-New York, 1979, viii + 418 pp., $19.50.

During a conversation in 1966, G. Köthe said to me, "The best book about the general theory of nuclear spaces and tensor products in existence today is the book of A. Pietsch [16]". Perhaps the kindest thing to be said about the present book is that Köthe's statement is still true. In fact, although he tries to hide it with new names such as "prenuclear norm", most of Wong's book could have been written at the time of that conversation. He doesn't write very much about what has happened in the ensuing 14 years.

It is common knowledge that the origin of the theory of nuclear spaces (and tensor products, too—there is little to be said about Schwartz Spaces these days) lies in the thesis of A. Grothendieck [10]. I'll indicate some