T. Mitchell [1966], *Fixed points and multiplicative left invariant means*, Trans. Amer. Math. Soc. **122**, 195–202.

______ [1970], *Topological semigroups and fixed points*, Illinois J. Math. **14**, 630–641.

J. von Neumann [1929], *Zur allgemeine Theorie des Masses*, Fund. Math. **13**, 73–116.

______ (1934), *Almost periodic functions in a group*. I, Trans. Amer. Math. Soc. **36**, 445–492.

T. Tannaka [1939], *Über den Dualitätssatz der nichtkommutativen topologischen Gruppen*, Tôhoku Math. J. **45**, 2–12.

B. L. van der Waerden [1933], *Stetigkeitssatze der halbeinfachen Lieschen Gruppen*, Math. Z. **36**, 780–786.

MAHLON M. DAY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, July 1980
© 1980 American Mathematical Society
0002-9904/80/0000-0313/$02.50

Von Neumann regular rings, by K. R. Goodearl, Monographs and Studies in Mathematics, No. 4, Pitman, London-San Francisco-Melbourne, 1979, xvii + 369 pp., $46.00.

The conception of von Neumann regular rings occurred in 1936 when John von Neumann defined a *regular ring* as a ring R with the property that for each $a \in R$ there exists $b \in R$ such that $a = aba$. In order to distinguish these rings from the regular Noetherian rings of commutative algebra, noncommutative ring theorists have added von Neumann's name as a modifier. There is, however, very little chance of confusing these two concepts since their only common objects of study would be fields. The standard example of a regular ring is the complete ring of linear transformations of a vector space over a division ring.

Motivated by the coordinatization of projective geometry which was being reworked at that time in terms of lattices, von Neumann introduced regular rings as an algebraic tool for studying certain lattices. The lattices von Neumann was interested in had arisen in joint work with F. J. Murray dealing with algebras of operators on a Hilbert space [10], which subsequently came to be known as *von Neumann algebras* or *W^*-algebras*. Although a W^*-algebra A turns out to be a regular ring only when A is finite-dimensional, a regular ring can be assigned to A by working with the set $P(A)$ of projections, a projection on A being a selfadjoint idempotent. For a finite W^*-algebra A, Murray and von Neumann used a regular ring R to "coordinatize" $P(A)$ in the sense that $P(A)$ turned out to be naturally isomorphic to the lattice of principal right ideals of R. (Finite means that $tt^* = 1$ whenever $t^*t = 1$, for $t \in A$.) Expanding on this idea [14], von Neumann invented regular rings so as to coordinatize complemented modular lattices, a lattice L being coordinatized by a regular ring R if it is isomorphic to the lattice of principal right ideals of R. As von Neumann showed, almost all complemented modular lattices could be coordinatized by a regular ring.

The roots of regular rings were firmly embedded in the theory of operator algebras and lattice theory. From the purely ring-theoretic viewpoint regular rings as a subject of investigation were largely ignored for a long period of time. In N. Jacobson's bible for ring theorists [5], regular rings are mentioned only briefly (p. 210). Yet there were intimations that regular rings might be worthy of study for their own sake, since they appeared in various contexts.

Regular rings are homologically characterized as those rings for which all modules (left or right) are flat [1], [4]. Accordingly, the Bourbakian school refers to regular rings as *absolutely flat* rings. Commutative regular rings may be characterized in many ways: (i) rings with no nonzero nilpotent element having all prime ideals maximal; (ii) rings for which all simple modules are injective [13]; (iii) rings for which localization at any maximal ideal yields a field [13]; (iv) the polynomial ring in one variable is semihereditary [2], [9]. Further, R. S. Pierce's 1967 Memoir [11], amply demonstrated the rich connection between the theory of sheaves and commutative regular rings.

On the noncommutative side, R. E. Johnson introduced nonsingular rings in 1951 [6] and maximal quotient rings were born. A submodule N of an R-module M is an *essential submodule* if $N \cap K \neq 0$ for all nonzero submodules K of M. A *nonsingular* right R-module M is a module for which the only element of M which is annihilated by an essential right ideal of R is the zero element. When the ring R itself is a nonsingular right R-module, Johnson showed that R could be embedded in a regular ring Q, the *maximal right quotient ring* of R, as an essential R-submodule of Q. In addition Q was injective as a right Q-module, thus was called a *right self-injective ring*, and injective as an R-module. Later Q came to be identified as the injective hull of the right R-module R with an appropriate ring multiplication extending the module multiplication. More importantly, Q was canonically isomorphic to the endomorphism ring $\mathrm{End}_R(Q)$, leading Johnson and Wong to establish that $\mathrm{End}_R(M)$ is always a right self-injective regular ring whenever M is a nonsingular injective right R-module [7]. If you should wonder whether there might be some connection between nonsingular rings, maximal quotient rings, finite W^*-algebras, and the regular rings used by Murray and von Neumann there is. The operator algebras under consideration were (left and right) nonsingular rings and the regular ring attached to their projection lattice was the maximal quotient ring, as J.-E. Roos observed [12]. Regular rings also made an appearance in category theory where P. Gabriel and U. Oberst [3] established that for a spectral category S with a generator U, the "endomorphism" ring R of U is a regular right self-injective ring.

Interest in regular rings accelerated during the late 60's and 70's, with many active contributors. Special mention should be made of the work of I. Halperin and Y. Utumi, both of whom studied regular rings which reflected the lattice-theoretic beginnings of the subject. Significant recent developments were made by G. Renault in the area of regular self-injective rings, while the penetrating work of D. Handelman and Goodearl constituted the foundation of a systematic theory of regular rings.

In view of the lively interest in them, as well as the many fine and probing results in existence about them, it is clear that regular rings were a subject waiting for a book to happen. Appropriately the task was taken on by someone who is an innovative scholar as well as a thorough meticulous expositor. The result is a well-crafted object which should delight any ring-theorist. At the same time, the non-ring-theorist who wishes to become acquainted with an area of ring theory that has possible ramifications outside of the field can do so and come away with a healthy understanding of the vitality of the subject matter.

The formal requirements for reading this book are not very stringent; anyone with a basic knowledge of projective modules and tensor products, for example, can do so. Direct and inverse limits are used, but these are (or should be) ideas with which all mathematicians are acquainted. While this is a book in the theory of rings, very little structure theory is required other than knowledge of the Wedderburn-Artin theorem, although the maximal quotient ring is mentioned and used throughout. Its introduction at the end of Chapter 1 is both innocuous and unpainful. I should add also that no knowledge of operator theory is required; the treatment and content is almost totally algebraic.

Goodearl has chosen to concentrate his study on the structure of noncommutative regular rings and their modules. The characterizations (i), (ii) and (iii), I mentioned previously of commutative rings are included (Theorem 1.16) but not dwelt upon. The choice of emphasis is understandable since the results, techniques and machinery used to examine commutative regular rings rarely extend to the noncommutative situation. As noted in the book's preface, von Neumann's coordinatization theorem is excluded on the basis that it is almost entirely a subject in lattice theory and too long for inclusion in a book on ring theory. What is presented are discussions of those classes of regular rings which have been most instrumental in the formulation of the theory; this is the subject matter of Chapters 3–14. In Chapters 15–18, tools for studying general regular rings are developed; Chapters 19–21 deals with some applications of these methods. Here is a bit more detail about what can be expected.

The first two chapters are introductory in nature. From the definition of a regular ring R, it immediately follows that if $a = aba$ then $ab = e$ is an idempotent and $eR = aR$. More can be deduced, namely that any finitely-generated right ideal of R is generated by an idempotent element and so is a direct summand. From here it is a short step to showing that $L(R)$, the set of finitely-generated right ideals of R, is a complemented modular lattice. A remarkable property of regular rings is that R may be replaced by any projective R-module A; that is, the set $L(A)$ of finitely generated submodules forms a modular lattice and is complemented when A is finitely generated. This rich lattice structure permits passage between R and the endomorphism ring of a finitely-generated projective module (it's also a regular ring). It also permits the derivation of results such as Theorem 2.8 which shows that for a finitely-generated projective module over a regular ring, any two direct sum decompositions have an isomorphic refinement.

Beginning with Chapter 3 various restrictions are placed on regular rings. Chapter 3 considers regular rings having all their idempotents in the center. These are called *abelian* regular rings. They reflect many of the properties of commutative regular rings such as having no nilpotent elements other than zero and all one-sided ideals being two-sided. The necessity of looking at these rings is that they occur in two notable situations later on. In Chapter 6 it is shown (Theorem 6.6) that if R is a regular ring all of whose primitive homomorphic images are Artinian and J is a nonzero ideal of R then J contains a nonzero central idempotent e such that eR is isomorphic to a full matrix ring over an abelian regular ring. In Chapter 13 right continuous regular rings are introduced. The motivation is lattice theoretic reflecting the continuous geometries of von Neumann. In terms of rings, however, they may be characterized as regular rings in which each right ideal is an essential submodule of a principal right ideal of R. It is easily checked that regular right self-injective rings are right continuous. The obstruction to the converse holding is precisely an abelian regular summand since any right continuous regular ring is a direct sum of an abelian continuous regular ring and a right self-injective ring.

If any two particular conditions on regular rings should be singled out they are unit-regularity and direct finiteness, the objects of study in Chapters 4 and 5, respectively. A regular ring R is *unit-regular* if for each $a \in R$ there is an invertible element $u \in R$ such that $a = aua$, while R is *directly finite* if $xy = 1$ whenever $yx = 1$. It is easily seen that unit-regular rings are directly finite. One of the highlights of Chapter 5 is the presentation of G. Bergman's example of a directly finite ring R which is not unit-regular; in fact all matrix rings over R are directly finite. Both unit-regular and direct-finiteness are weak forms of finiteness conditions for regular rings and, as such, lead to results which are reminiscent of the full ring of linear transformations of a finite-dimensional vector space over a division ring. An example of Such a property is the cancellation characterization of unit-regularity given in Theorem 4.5: *A regular ring R is unit-regular if and only if $A \oplus B \approx A \oplus C$ implies $B \approx C$ for all finitely-generated projective right R-modules.* On the other hand, direct-finiteness is used in the classification of regular right self-injective rings carried out in Chapter 10. Here the theory of types, originally proposed by Murray and von Neumann, and later developed as a classification scheme for Baer rings by Kaplansky [8] is developed. Since regular right-self injective rings are Baer rings, every regular right self-injective ring R is uniquely a direct product of rings of Types I_f, I_∞, II_f, II_∞, III. The meaning of these symbols is a bit too technical to reproduce here but the central theme revolves on the existence of certain idempotents e for which the subring eRe possesses properties such as abelian or direct-finiteness. One of these types is easy to describe: *A regular right self-injective ring is of Type I_f if and only if $R \approx \Pi_{n>1} R_n$, where each R_n is an $n \times n$ matrix ring over an abelian regular ring* (Theorem 10.24). Some curious things occur with the other types; for example, a regular right self-injective ring is Type III if and only if $xR \oplus xR \approx xR$ for all $x \in R$ (Corollary 10.17).

Dimension functions are discussed in Chapters 11 and 12. These are functions defined on the nonsingular injective modules over a regular self-injective ring and are of two kinds, relative dimension functions and infinite dimension functions. These determine the isomorphism classes of certain nonsingular injective modules. These dimension functions are indexed by the maximal ideals of the Boolean algebra $B(R)$ of central idempotents of R. When R is prime, $B(R) = \{0, 1\}$ and there is only one infinite dimension function. The most striking application of the theory developed in this chapter is the use of this dimension function to show that in a prime regular right self-injective ring the two-sided ideals are well ordered by inclusion (Theorem 12.23).

Continuous rings and $\aleph_0$-continuous rings can be thought of as weak forms of injectivity. Accordingly much of Chapter 13 involves connections between continuous rings and right self-injective rings. As with right continuous rings, $\aleph_0$-continuity can be phrased in terms of right ideals; i.e. a regular ring R is right $\aleph_0$-*continuous* if and only if each countably-generated right ideal is essential in a principal right ideal. If any one result exemplifies the methods and techniques which are required to study regular rings, it is Theorem 14.24: *Every left and right $\aleph_0$-continuous regular ring is unit-regular.*

Having sampled the chapters dealing with regular rings satisfying additional hypotheses what does the author propose as tools for the study of regular rings in general and how useful might these be? These two questions are answered quite well in the latter portions of the book. Here we find the Grothendieck group $K_0(R)$ of R, rank and pseudo-rank functions, and completions relative to pseudo-metrics induced by pseudo-rank functions. Examples of K_0 are constructed for various types of regular rings such as abelian regular or regular right self-injective of Type II_f, etc. That K_0 can indeed be a powerful tool in the study of regular rings is demonstrated by two results which I cite.

COROLLARY 15.21. *For a unit-regular ring R the lattice $L_2(R)$ of two sided ideals of R is isomorphic to the lattice of directed convex subgroups of $K_0(R)$.*

An algebra R over a field F is *ultramatricial* if R is a union of an ascending sequence $R_1 \subseteq R_2 \subseteq \cdots$ of F-algebras each of which is a finite direct sum of full matrix rings over F. For an ultramatricial algebra R, $K_0(R)$ has as order unit, the equivalence class $[R]$ determined by R. For ultramatricial algebras R and S Theorem 15.26 shows that $R \approx S$ *as F-algebras whenever* $K_0(R, [R]) \approx K_0(S, [S])$ *as directed groups.*

Rank and pseudo-rank functions on a regular ring are suitable generalizations of normalized matrix rank (i.e. divide the matrix rank by the size of the matrix). In the case of a Boolean ring R, a pseudo-rank function turns out to be a nonnegative, normalized, finitely additive measure which is a rank function if and only if it is strictly positive. The set of all pseudo-rank functions on R is denoted by $\mathbf{P}(R)$. Their existence and/or uniqueness is examined in Chapter 18 and Theorem 18.3 gives a criterion for existence namely, $\mathbf{P}(R)$ is not empty if and only if R has a proper two-sided ideal K such that all matrix rings over R/K are directly finite. Consequently, in examining the question of the uniqueness of a pseudo-rank function on R, one sees the hypothesis that R is a regular ring all of whose matrix rings are directly finite. It is worthwhile to mention here that the problem of whether or not $n \times n$ matrices over directly finite regular rings are directly finite probably ranks as the number 1 open question in regular rings. When $\mathbf{P}(R)$ is nonempty it turns out to be a Choquet simplex (Theorem 17.5). This geometric structure is the main reason that $\mathbf{P}(R)$ succeeds in serving as a tool for studying regular rings. Goodearl has been kind enough to include an appendix on compact convex sets to help in the understanding of this matter. Finally, completions in the pseudo-metrics induced by subsets of $\mathbf{P}(R)$ are examined vis-a-vis self-injectivity. The completion relative to such a family is a unit-regular left and right self-injective ring. Naturally, one wishes to know when self-injective regular rings are complete and this occurs if and only if

$B(R)$ is complete with respect to the family (Theorem 21.16).

These brief comments should provide a flavor of the topics covered. Three additional things are worth mentioning. The bibliographic notes at the end of each chapter provide attributions of the various results included in the chapter and serve as a superb guide to the literature. Those interested in testing their mastery of the subject can have a go at the 57 open problems which have been assembled. They should be aware, however, that Handelman has answered #23 in the affirmative and P. Menal has given a positive answer to #53. Perhaps the strongest point of this monograph is the inclusion of many diverse and interesting examples. The book abounds in these and their presentation makes the subject lively and interesting.

The book is not without a bit of imperfection here and there. The statement at the bottom of page 20, that every regular ring can be expressed as a direct union of right and left hereditary regular subrings is not an immediate consequence of what has gone before as the reader might be led to expect. The matter becomes clear after one sees the construction in Example 5.18 (page 58), however. At the beginning of Chapter 3 the assertion that "a nontrivial theorem is required to show that strongly regular rings are regular" is debatable. (R is *strongly regular* if for each $a \in R$ there exists $b \in R$ with $a = a^2b$.) But to mention any other such occurrences would border on pettiness. This is an important piece of work and it is one to which I will be returning repeatedly. I suspect quite a number of present-day and future ring-theorists will do so as well.

References

1. M. Auslander, *On regular group rings*, Proc. Amer. Math. Soc. **8** (1957), 658–664.
2. V. Camillo, *Semihereditary polynomial rings*, Proc. Amer. Math. Soc. **45** (1974), 173–174.
3. P. Gabriel and U. Oberst, *Spektralkategorien und regulare Ringe im Von-Neumannschen Sinn*, Math. Z. **92** (1964), 389–395.
4. M. Harada, *Note on the dimension of modules and algebras*, J. Inst. Poly. Osaka City Univ. **7** (1956), 17–27.
5. N. Jacobson, *Structure of rings*, (2nd ed.), Amer. Math. Soc. Colloq. Publ., no. 37, Amer. Math. Soc., Providence, R. I., 1964.
6. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.
7. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc. **36** (1961), 260–268.
8. I. Kaplansky, *Rings of operators*, Benjamin, New York, 1968.
9. P. J. McCarthy, *The ring of polynomials over a von Neumann regular ring*, Proc. Amer. Math. Soc. **39** (1973), 253–254.
10. F. J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.
11. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70 (1967).
12. J.-E. Roos, *Sur l'anneau maximal des fractions des AW^*-algebras*, C. R. Acad. Sci. Paris Ser. A-B **266** (1968), 120–123.
13. A. Rosenberg and D. Zelinsky, *Finiteness of the injective hull*, Math. Z. **70** (1959), 372–380.
14. J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 707–713.

EFRAIM P. ARMENDARIZ