are integrable sets. Also the integral μ extends naturally to $L^1(\mu)$ and thus yields a measure on the integrable sets. This leads to a full discussion of the Lebesgue convergence theorems and Fubini's theorem, which complete Chapter 5.

In addition to the material discussed above, Bridges gives a fairly general version of the Stone-Weierstrass theorem in Chapter 4 and treats the functional calculus for bounded, selfadjoint operators on Hilbert space in Chapter 6. He also gives an extensive list of references which will be useful to anyone who wishes to see what a wide variety of constructive mathematics, and not just in analysis, has been developed since the appearance of Bishop's book.

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Discontinuous Čebyšev systems, by Roland Zielke, Lecture Notes in Math., vol. 707, Springer-Verlag, Berlin-Heidelberg-New York, 1979, vi + 111 pp., $9.00.

A finite set of real-valued functions $g_1, \ldots, g_n$ having a common domain is linearly independent if and only if there exists a set of points $x_1, \ldots, x_n$ for which the determinant $\det(g_i(x_j))$ is nonzero. On the other hand, if this determinant is nonzero for *all* choices of distinct points $x_1, \ldots, x_n$, then the functions are said to comprise a *generalized Tchebycheff system* (GTS). Equivalently, one says that each nontrivial linear combination of the functions can have at most $n - 1$ zeros. Thus the concept of a GTS arises naturally by abstracting one important property of the monomial functions $1, x, x^2, \ldots, x^{n-1}$.

In approximation theory, the GTS emerges as a suitable mechanism for interpolation and approximation with various norms. For example, a polynomial of degree at most $n - 1$ can always be found taking prescribed values at n distinct points. But the same is true for the linear combinations of any GTS of order n, and indeed this property too could have served as the definition. Somewhat more recondite is the theorem of Tchebycheff [1859]: Each continuous function f defined on a compact interval $[a, b]$ possesses a unique best uniform approximation by a polynomial of degree at most $n - 1$: i.e., a polynomial p such that the expression

$$\|f - p\| = \max_{a \leq x \leq b} |f(x) - p(x)|$$

is a minimum. The polynomial is completely characterized by the existence of $n+1$ points x_i satisfying $a \leq x_0 < \cdots < x_n \leq b$ and $f(x_i) - p(x_i) = \sigma_i \| f - p \|$, with $\sigma_i \sigma_{i-1} = -1$. Again, this theorem remains true if the set of monomials $1, \ldots, x^{n-1}$ is replaced by any continuous GTS of order n on $[a, b]$.

In 1914, Alfred Haar proved a more surprising result–one which suggests that the GTS provides the "correct" setting for uniform approximation: In order that every continuous function on a compact space shall have a unique best approximation of the form $\Sigma_1^n \lambda_i g_i$, where the functions g_i are continuous and form a linearly independent set, it is necessary and sufficient that the set be a GTS.

Haar observed at the same time that the existence of a continuous GTS of order 2 or more implied that the domain could not contain a homeomorph of the 2-sphere. (On S^2, one can interchange the positions of two points with a continuous motion during which the points never coincide. In this process, $\det(g_i(x_j))$ will pass through the value 0.) This direction of inquiry eventually led in 1956 to the theorem of Mairhuber and others that a continuous GTS of order 2 or more exists on a compact space only if the space can be homeomorphically imbedded in a circle. R. Zielke has made important contributions to this topic, and his lecture notes under review give probably the best available account of it.

In the theory of best approximation using norms other than the uniform norm, the GTS still plays a rôle. Dunham Jackson proved in 1921 that if f is a continuous function on $[a, b]$ and if $\{g_1, \ldots, g_n\}$ is a continuous GTS, then a unique set of coefficients λ_i exists to make the expression $\int |f - \Sigma \lambda_i g_i|$ a minimum. This result appears more remarkable in light of the fact that within the space $L^1[a, b]$, *no* finite-dimensional subspace can provide unique best approximations to all functions.

While the GTS has played a continual rôle in approximation theory from the time of Tchebycheff, its pervasive influence in other branches of mathematics went largely unnoticed until the appearance in 1966 of the remarkable treatise of S. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics*, Interscience Publishers, New York, MR **34** 4757. This tour de force of approximately 600 pages begins with a succinct account of the theory of these systems and of related systems such as "weak" and "extended complete" Tchebycheff systems. Then in 13 subsequent chapters, a wealth of applications is presented in such fields as interpolation, moment problems, boundary value problems, theory of inequalities, and convexity.

Karlin and Studden consider principally the continuous GTS on a compact interval. In accordance with traditional terminology these are termed simply Tchebycheff systems. Also important are the weak Tchebycheff systems; these are defined by requiring only $\det(g_i(x_j)) \geq 0$ whenever $x_1 < \cdots < x_n$.

After assuming continuity, it is not much of a restriction to assume further that the domain is a subset of the reals, because of Mairhuber's Theorem. But there certainly exist situations in which a theory with weaker postulates is wanted. Zielke has succeeded well in providing such a theory in this monograph, and he has also gone to some trouble to improve vintage results wherever he could. He has obviously kept foremost in his mind the possible applications in approximation theory. A large part of the book is devoted to the following three central problems. In each case the problem can be posed in terms of Tchebycheff systems, generalized Tchebycheff systems, or weak Tchebycheff systems. I. If a GTS is given, does it contain a GTS of order one less? II. If G is a GTS, does there exist a function f such that $G \cup \{f\}$ is a GTS? III. If a function f and an n are given, does there exist a GTS of order n containing f?

For a hint as to how such questions arise, let us cite a theorem of Krein: If $\{1, x, \ldots, x^n, f\}$ is a Tchebycheff system on $[-1, 1]$, then the polynomial p of degree at most n which minimizes $\int_{-1}^{1}|f - p|$ is the polynomial which interpolates to f at the points $\cos k\pi/(n + 2)$, $1 \leq k \leq n + 1$.

The problems mentioned above do not have clear-cut answers in all cases, and work on them continues. Zielke's account of the subject is therefore not final, but it is nevertheless a valuable summary of the current status.

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Applied mathematics: An intellectual orientation, by Francis J. Murray, Mathematical Concepts and Methods in Science and Engineering, Volume 12, Plenum Press, New York-London, 1978, xiv + 255 pp.

It is essential from time to time, as the academic world revolves, and as each revolution carries us to new heights of specialization, to refresh our understanding of relationships among disciplines. What has history to do with psychoanalysis, music with computer science, economics with ecology, language with linguistics? It may also be useful on suitable occasions to ask ourselves what a given discipline actually *is* in the contemporary academic context. Professor Murray, Director of Special Research on Numerical Analysis at Duke University, has produced a book that can be regarded as the mark of such an occasion. What, in the rising clamor of academic voices fighting to be heard, *is* Applied Mathematics? Then, having done our best with that, we can examine the relationship forming a central theme of Murray's book. What has mathematics to do with physics? The questions themselves, entirely aside from the character of our answers tend to raise red flags among pure mathematicians. The prospect of finding today's theorem in the design of tomorrow's missile system, or even in next year's solar engines, is discordant with what has become the conventional view of academic mathematics. Here the strongest work is the most abstract and, *a fortiori*, application is evidence of weakness. It may not be unfair to express this view in paraphrase of a remark by Clemenceau: applied mathematics bears the relation to mathematics that military music bears to music.

Readers of the history of mathematics need not be reminded that the growth of support for such attitudes among the majority of our contemporaries–there are, of course, a few virtuoso mathematicians who practice and defend the longer tradition–is recent and swift. To ask for a definition of useful mathematics would have been as puzzling to our academic forebears as