seems unusually readable and has the additional attraction, which devotees of the subject have learned not to take for granted, that the separation theorem is correctly stated.

The last two chapters treat subjects which are "nonstandard" in one sense or another. In Chapter 8 the authors discuss unbounded control input elements and sensing functionals. This subject has no finite dimensional counterpart but is "of the essence" in discussing control and observation of processes described by partial differential equations where control variables appearing in the boundary conditions and state measurements made at boundary points are important both because they are physically the most realizable and because, mathematically, they tend to provide the strongest controllability and observability results. The treatment is admirably general from the abstract point of view but appears, at first reading to be limited in application to analytic semigroups. This is certainly forgivable since the corresponding theory presented in a context wide enough to include, e.g. hyperbolic systems, would necessarily be very complicated and would have to resort to description of a number of special cases. Chapter 9 is a discussion of time dependent infinite dimensional systems carried out in the context of quasievolution operators and generators This material is nonstandard to the degree that in the differential equations and functional analysis literature generally, the extension to infinite dimensional systems of the very complete theory of finite dimensional linear systems with time varying coefficients is fraught with all sorts of technical difficulties. The main objective of this section is the study of the linear-quadratic optimal control problem (linear system, quadratic objective function) for a time-varying infinite dimensional system on a finite time interval.

The book has ample lists of references, one following each chapter and a long supplementary list at the end. I do feel that more commentary on the nature of these referenced contributions in the body of the text would have been helpful but this is mere quibbling. The book is a very sound one and the authors are to be congratulated on a significant achievement.

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Constructive functional analysis, by D. S. Bridges, Research Notes in Mathematics, Volume 28, Pitman, London-San Francisco-Melbourne, 1979, vi + 203 pp., $15.00.

The appearance in 1967 of Errett Bishop's book *Foundations of constructive analysis* was a significant event. Until then it had been a commonplace that the constructive point of view toward mathematical truth could be successful only in a few areas of mathematics, and certainly not in analysis or topology. That is, it was believed that these areas would inevitably be trivial and uninteresting if constructive principles were followed. Errett Bishop deflated this opinion by showing explicitly how to develop a substantial portion of abstract analysis constructively. Moreover, the subject matter and style of

Bishop's treatment made his work immediately accessible to any modern mathematician, even one who might reject the restrictions on mathematical reasoning which Bishop followed. The same is true of the wide variety of work which has been done under Bishop's influence since 1967, including the book under review.

Mathematicians have long been interested in constructivity and have placed a high value on achieving it. In the aesthetics of mathematics, constructivity competes with such other values as elegance, simplicity, clarity, etc. These are not "extras", but important elements of mathematical knowledge. Constructivity is often closely associated with deepening of understanding, with the answering of questions which occur naturally to the intellectually curious mathematician: how does an object, proved in some way to exist, depend on initial data? how can one compute its various numerical invariants? how closely can it be approximated? how explicitly can it be described?

In subjects such as number theory or combinatorics, where the objects studied are "finite" in their natural form, constructivity is widely found. What Bishop did was to show, by detailed example, how to proceed constructively in analysis and related subjects, where it is less obvious that anything systematic can be achieved. Of course Bishop himself excludes all forms of reasoning which do not satisfy his quite strict standards of constructivity; for him the statements of mathematics are meaningless except on a constructive interpretation; proofs are unconvincing unless constructive. Moreover, he often takes a polemical tone and interlaces his quite interesting mathematical ideas with harsh characterizations of those who take a viewpoint different from his own. However these usually seem not to be meant as a serious part of the discussion (as when, for example, Bishop characterizes classical mathematicians as avoiding the "axiom" $0 = 1$ only because to do otherwise would put them out of business). Any mathematician will find that he can enjoy the mathematics done by Bishop and his "school" and can learn from it, without accepting their philosophical dogma. Moreover, a careful reading of this work will deepen his own view of mathematics and will assist him in making sharper and more delicate distinctions of meaning in his own work.

Bishop's book is unfortunately no longer available and the book under review is meant to provide continued access to the constructive functional analysis which Bishop developed. Bridges has omitted a few topics (complex analysis, locally compact abelian groups, ergodic theory) which were treated by Bishop and focuses his attention on the core of functional analysis: the topology of metric spaces, the Hahn-Banach theorem and duality for normed linear spaces, the Stone-Weierstrass theorem, integration on locally compact metric spaces, and the functional calculus for bounded, selfadjoint operators on Hilbert space. Within this subject matter Bridges follows Bishop rather closely, giving some results in a more general form and proving others more smoothly by taking advantage of developments since 1967.

The starting point for any constructive theory is to ask what it means to give or to present a mathematical object of some type. For integers or rational numbers or for combinatorial objects the answer is clear. Even for the real numbers, at the beginning of analysis, the answer is not difficult: to give a real number is to give a procedure which will accept any positive integer n and will produce a rational number x_n which is an approximation of the desired real number to within an error of $1/n$. The procedure must be deterministic, a finite collection of rules by which x_n is obtained from n. Moreover we must have in hand a proof that the sequence (x_n) carries out the desired approximation; this amounts to a proof that for each m and n

$$|x_m - x_n| < \frac{1}{m} + \frac{1}{n}.$$

We then think of the real number as being the sequence (x_n). One could just as well take a real number to be any sequence of rational numbers which is given by some explicit procedure and for which one has an explicit proof that it is a Cauchy sequence. The adoption of a specific rate of convergence is a matter of convenience, whether one is thinking constructively or not. Note that we need not exclude numbers which we cannot give in this explicit way, although Bishop does do so. Rather we can think of this as setting a requirement which must be met if we are to prove the existence of a number constructively. Each time a new type of mathematical object is introduced one must carry out a similar discussion.

Taking a constructive point of view leads one to make distinctions and choices which would not otherwise arise and to reexamine known proofs, with results which only serve to make mathematics deeper and richer. As an elementary example, consider the Intermediate Value Theorem (IVT), one of the cornerstones of basic analysis, in the following form: if f: $[0, 1] \rightarrow R$ is uniformly continuous and if $f(0) < y$ and $f(1) > y$, then there exists x in $[0, 1]$ such that $f(x) = y$. Suppose that we wish to obtain x constructively from y and f. The usual proofs do not accomplish this. For example, consider the familiar "interval-chopping" procedure for approximating x. Setting $a_1 = 0$ and $b_1 = 1$, we generate sequences of rational numbers a_n, b_n by induction so that the lengths of the intervals $[a_n, b_n]$ go to 0 very fast and also so that $f(a_n) < y$ and $f(b_n) > y$ for all n. Then (a_n) gives the desired x such that $f(x) = y$. The induction step consists of letting $[a_{n+1}, b_{n+1}]$ equal either $[a_n, d]$ or $[d, b_n]$, where d is the midpoint of $[a_n, b_n]$ and the choice of which interval to use depends on whether $f(d) > y$ or $f(d) < y$. Of course if $f(d) = y$ then we need not continue.

While this certainly has a very constructive flavor, there is a problem: we have no constructive procedure for deciding whether $f(d)$ is $>y$, $<y$ or $=y$. Indeed, it is unlikely (in the extreme) that we will ever find such a procedure, as can be argued using what Bishop calls a "counterexample in the style of Brouwer." This is not really a counterexample at all, but a demonstration of how to construct a procedure for systematically solving a large class of difficult unsolved mathematical problems, given (in this case) a hypothetical procedure for deciding the ordering of pairs of real numbers. For instance, given a sequence (α_n) of 0's and 1's, let z be the real number whose mth rational approximation has the binary representation

$$z_m = .\alpha_1\alpha_2 \ldots \alpha_m.$$

Then a constructive procedure for deciding whether a real number is positive will, when applied to z, yield either a specific n for which $\alpha_n = 1$ or the information that $\alpha_n = 0$ for all n. Clearly we do not have any such procedure at present, nor do we have any reasonable expectation of finding one soon.

As often happens, the constructive "difficulty" described above corresponds to problems which arise in practice when one computes using calculating devices. When finding roots of f by interval halving one may reach a point where the value of $f(d)$ calculated is indistinguishable from 0 within the calculating device. Yet one cannot infer that this value of d is near to a root of f.

The constructive version of the IVT which Bridges gives has the added hypothesis that y is unequal to $f(d)$ for each rational number d in $[0, 1]$. By this is meant that one has a procedure which yields for each such d an integer m such that $|f(d) - y| > 1/m$. When this hypothesis holds, then one can carry out the decisions required in the interval halving process and thus make it constructive.

This is only one of many constructive versions of the IVT. This version is not really very satisfactory when one is dealing with specific functions. For instance, if we use this result to prove constructively that $\sin(x) = 1/2$ has a solution in $[0, 1]$, then we must first prove that $\sin(d)$ is unequal to $1/2$ for each rational d in $[0, 1]$. Another version of the IVT, not mentioned by Bridges but presented by Bishop in [3], allows y to be arbitrary and restricts f. It corresponds to a modification of the interval halving procedure to allow replacing the midpoint of $[a_n, b_n]$ by a nearby point d such that $f(d)$ is unequal to y. The required assumption on f is that one has a procedure which, given y, n and z, will produce d and m such that $|d - z| < 1/n$ and $|f(d) - y| > 1/m$. This condition is satisfied by many elementary functions, such as polynomials, $\sin(x)$, $\exp(x)$, etc. and therefore the unrestricted IVT is valid for them.

The various constructive versions of the IVT correspond to the many different proofs one might give for the one classical result. In general the constructive mathematician sees a theorem not as a simple statement of fact, but as such a statement together with a proof (or proofs). This point of view, which certainly need not be restricted to the constructive domain, leads to a healthy emphasis on the inter-relatedness of mathematics, to an essentially scholarly attitude. In Bishop's book especially, one finds very extensive discussion of the choices he has made of how to formulate concepts and which results to present, of what areas have good constructive content. Unfortunately Bridges stays very close to the mathematical details and only briefly reflects on such matters. The reader will have to look for them to Bishop's book and to his articles [2], [3], [4] and [5].

The heart of Bridges' book lies in Chapters 3 and 5. Chapter 3 treats normed linear spaces, with the Hahn-Banach theorem as its centerpiece. Chapter 5 covers integration on locally compact metric spaces. The Hahn-Banach theorem, as usually proved, seems to have constructive content only in the finite-dimensional setting. Yet Bishop was able to find the following very general constructive version: let E be a separable normed space and F a linear subspace of E. Let u be a nonzero linear functional on F which is normable (in the sense that one has a procedure for computing the norm of u). Then for each $\varepsilon > 0$ one can construct a normable linear functional $u^{\#}$ on

E such that $u^{\#}(x) = u(x)$ for each x in F and

$$\|u^{\#}\| \leq \|u\| + \varepsilon.$$

As a consequence, given any x in E and any $\varepsilon > 0$, one can construct a normable linear functional u on E such that $\|u\| = 1$ and $u(x) > \|x\| - \varepsilon$.

The assumption that E is separable is important not only in the Hahn-Banach theorem but also in the treatment of the dual space E' of E. This is not a normed space in the constructive setting, since the assumption that all bounded linear functionals are normable is not justified. However, when E is separable then the weak* topology on E' is metrizable, on bounded sets. It is certain metrics which give this topology on B', the closed unit ball of E', which play the fundamental role in the duality theory. For example, one can prove constructively that B' is compact in the weak* topology. This requires giving, for each $\varepsilon > 0$, a specific finite set of linear functionals which is ε-dense in B' relative to a metric for the weak* topology. Moreover one has a constructive version of the fact that E is (if complete) the space of weak* continuous linear functionals on its dual space: if ϕ is a linear functional on E' and if ϕ is uniformly continuous on B' relative to a metric which gives the weak* topology, then for each $\varepsilon > 0$, one can construct an element x of E such that $|\phi(x') - x'(x)| < \varepsilon$ holds for every linear functional x' in B'.

Bridges' treatment of integration theory has benefited from work done since 1967 by Bishop, Henry Cheng and Y. K. Chan. The fundamental objects here are the integrals themselves; measures appear only after some effort has been applied. Let X be a locally compact metric space and let $C(X)$ be the linear space of all uniformly continuous, real-valued functions on X which have compact support. An *integral* on X is a positive linear-functional μ on $C(X)$. Elements of the space $L^1(\mu)$ are sequences (f_n) from $C(X)$ which satisfy the condition $\sum \mu(|f_n|) < \infty$. This sequence can be identified with a function f defined by

$$f(x) = \sum f_n(x)$$

on the set of x for which this series converges absolutely. Such an element of $L^1(\mu)$ is equal to 0 (in other words, "f is equal to 0 almost everywhere") when there is another sequence (g_n) in $C(X)$ such that $\sum \mu(|g_n|) < \infty$ and such that one has

$$\sum |f_n(x)| < \infty \quad \text{and} \quad \sum f_n(x) = 0 \quad \text{whenever} \quad \sum |g_n(x)| < \infty.$$

Corresponding to sets of finite measure are the *integrable* sets: these are pairs (A_0, A_1) of subsets of X such that $d(x_0, x_1) > 0$ constructively whenever x_0 is in A_0 and x_1 is in A_1 and such that the function f defined to be 1 on A_1 and 0 on A_0 is in $L^1(\mu)$. One major difficulty in this area is in showing that many integrable sets can be constructed. A major result is that for each f in $L^1(\mu)$ one can give a sequence (r_n) such that for each real number r which is distinct from every r_n, the pairs

$$(\{x | f(x) \geq r\}, \{x | f(x) < r\})$$

and

$$(\{x | f(x) > r\}, \{x | f(x) \leq r\})$$

are integrable sets. Also the integral μ extends naturally to $L^1(\mu)$ and thus yields a measure on the integrable sets. This leads to a full discussion of the Lebesgue convergence theorems and Fubini's theorem, which complete Chapter 5.

In addition to the material discussed above, Bridges gives a fairly general version of the Stone-Weierstrass theorem in Chapter 4 and treats the functional calculus for bounded, selfadjoint operators on Hilbert space in Chapter 6. He also gives an extensive list of references which will be useful to anyone who wishes to see what a wide variety of constructive mathematics, and not just in analysis, has been developed since the appearance of Bishop's book.

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Discontinuous Čebyšev systems, by Roland Zielke, Lecture Notes in Math., vol. 707, Springer-Verlag, Berlin-Heidelberg-New York, 1979, vi + 111 pp., $9.00.

A finite set of real-valued functions $g_1, \ldots, g_n$ having a common domain is linearly independent if and only if there exists a set of points $x_1, \ldots, x_n$ for which the determinant $\det(g_i(x_j))$ is nonzero. On the other hand, if this determinant is nonzero for *all* choices of distinct points $x_1, \ldots, x_n$, then the functions are said to comprise a *generalized Tchebycheff system* (GTS). Equivalently, one says that each nontrivial linear combination of the functions can have at most $n - 1$ zeros. Thus the concept of a GTS arises naturally by abstracting one important property of the monomial functions $1, x, x^2, \ldots, x^{n-1}$.

In approximation theory, the GTS emerges as a suitable mechanism for interpolation and approximation with various norms. For example, a polynomial of degree at most $n - 1$ can always be found taking prescribed values at n distinct points. But the same is true for the linear combinations of any GTS of order n, and indeed this property too could have served as the definition. Somewhat more recondite is the theorem of Tchebycheff [1859]: Each continuous function f defined on a compact interval $[a, b]$ possesses a unique best uniform approximation by a polynomial of degree at most $n - 1$: i.e., a polynomial p such that the expression

$$\|f - p\| = \max_{a \leqslant x \leqslant b} |f(x) - p(x)|$$