BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 1, July 1980

RESEARCH ANNOUNCEMENTS

ON THE ABUNDANCE OF APERIODIC BEHAVIOUR FOR MAPS ON THE UNIT INTERVAL[1]

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Continuous maps from the interval [0, 1] to itself have been studied for some time as simple models of dynamical systems with discrete time. In particular, the map $x \longmapsto 1 - 2|x - ½|$ has no stable periodic orbit on [0, 1]. In the paper [1] we show that such behaviour is very common among the members of a parametrized family of maps which contain a quadratic critical point.

Let $0 < \delta < ½$ and define the map f_δ: $[0, 1] \longrightarrow [0, 1]$ by

$$f_\delta(x) = \begin{cases} 1 - \delta - (x - ½)^2/\delta & \text{for } x \in [½ - \delta, ½ + \delta], \\ 1 - 2|x - ½| & \text{for } x \in [0, ½ - \delta] \cup [½ + \delta, 1]. \end{cases}$$

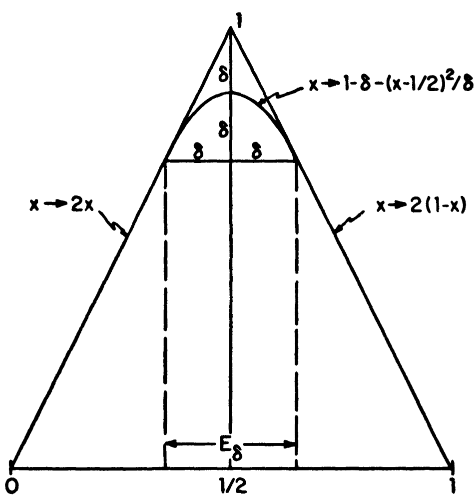

FIGURE 1. The function f_δ for $\delta = 0.15$.

Received by the editors September 7, 1979 and, in revised form, December 6, 1979.
AMS (MOS) subject classifications (1970). Primary 58F99; Secondary 28A65.
[1] This research was supported in part by NSF grant PHY-77-18762 and by the Fonds National Suisse.

In [1] we construct a Lebesgue measurable subset M of $[0, \frac{1}{2}]$ with positive measure having the properties:

(a) M gets thicker near $\delta = 0$, i.e., Leb. meas. $(M \cap [0, \delta])/\delta > 1 - 1/|\log \delta|$, for small $\delta > 0$.

(b) If $\delta \in M$, the map f_δ has no stable periodic orbit.

(c) If $\delta \in M, f_\delta$ is topologically conjugate to a piecewise linear map $g_{\tau(\delta)}$: $x \longrightarrow \tau(\delta) \cdot (1/2 - |x - \frac{1}{2}|)$, with $\tau > 2^{1/2}$.

REMARKS. (1) (a), (b), and (c) show the abundance of aperiodic behaviour among the f_δ, when δ is near 0. This settles an old question about maps on the interval.

(2) (c) implies sensitive dependence on initial conditions in the sense of Guckenheimer [2].

(3) The often-studied maps $x \longmapsto 4sx(1 - x)$ with $s = 1 - \delta^2\pi^2/8$ are conjugated (through $x = \sin^2(\pi y/2)$) to $y \longmapsto 1 - \delta - (2/\delta)(y - \frac{1}{2})^2 + O(\delta^2)$ (for y near $\frac{1}{2}$). This motivates our choice of f_δ.

SKETCH OF THE PROOF FOR THEOREM 1. The proof relies on a representation of a number in the interval $E_\delta = [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ by a sequence $(n_i, A_i, \epsilon_i)_{i \in N}$, where $n_i \in \mathrm{N}$, $A_i \in \mathrm{N}$ and $\epsilon_i = \pm 1$. The representation is defined recursively. Let $x_0 \in E_\delta$ be the number to be represented, and assume $(n_i, A_i, \epsilon_i)_{i=1,\dots,j}$ have been computed together with $x_j \in E_\delta$. n_{j+1} is defined as the smallest integer such that $f_\delta^{n_{j+1}}(x_j)$ is in E_δ. x_{j+1} is equal to $f_\delta^{n_{j+1}}(x_j)$, A_{j+1} is the integer part of

$$2^{n_{j+1}-1}(\delta + (x_j - \tfrac{1}{2})^2/\delta) \quad \text{and} \quad 2^{n_{j+1}-1}(\delta + (x_j - \tfrac{1}{2})^2/\delta) - A_{j+1}$$

is equal to $\epsilon_{j+1}x_{j+1}$. Some parts of the proof of Theorem 1 are reminiscent of the small divisor problem. In particular, if for some δ, $x_i = \frac{1}{2}$, then $\frac{1}{2}$ is a stable periodic point. Such a δ, together with a small δ interval around it, does not belong to M and can be considered as a resonance. The measure of M is investigated using a lower bound on $dx_i/d\delta$ when δ is in M.

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