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Lectures on von Neumann algebras, by Serban Strătilă and László Zsidó, Editura Academiei, Bucureşti, România, and Abacus Press, Turnbridge Wells, Kent, England, 1979, 478 pp.

An $n \times n$ matrix algebra M_n over the complex numbers $\mathbf{C}$ aleady exhibits some of the properties of a von Neumann algebra. This algebra acts naturally on the n-dimensional vector space $\mathbf{C}^n$. It is its own double commutant in the space of linear endomorphisms on $\mathbf{C}^n$. Here the commutant of a set S of endomorphisms is the set of all endomorphisms x such that $xs = sx$ for all s in S. In addition, the algebra M_n is the dual space of the space of linear functionals on M_n generated by the evaluation of the dot product $x \to x\alpha \cdot \beta$ $(\alpha, \beta \in \mathbf{C}^n)$. A distinguished functional on M_n is the trace ϕ. It takes a matrix into the sum of its diagonal elements and it is the unique functional satisfying $\phi(1) = 1$ and $\phi(xy) = \phi(yx)$ for all x, y in M_n. Then M_n is itself an inner product space with $\langle x, y\rangle = \phi(y^*x)$. Here y^* denotes the image of y under the involution equal to the transpose of the complex conjugate. The algebra M_n acts naturally on the inner product space by left multiplication. The commutant of M_n is the action of M_n by right multiplication so that M_n is its own double commutant. Again M_n is the dual space of the set of functionals on M_n defined by $x \to \langle xy, z\rangle$ $(y, z \in M_n)$.

Individual elements of M_n are also important. Distinguished among these are the selfadjoint elements $(x = x^*)$, the positive elements $(x = y^*y)$, and the projections $(p = p^* = p^2)$. A partial ordering exists for selfadjoint elements, viz. $x \geqslant y$ if $x - y$ is positive and an equivalence relation exists for projections, viz. $p \sim q$ if there is a v in M_n with $v^*v = p$, $vv^* = q$. So two projections in M_n are equivalent if and only if they have the same trace or if and only if the subspace $p(\mathbf{C}^n)$ and $q(\mathbf{C}^n)$ or the subspaces $p(M_n)$ and $q(M_n)$ (with M_n as an inner product space) have the same dimension. Furthermore, given any projections p and q in M_n then either $p \prec q$ (i.e. p is equivalent to a projection $q' \leqslant q$) or $q \prec p$.

All of these concepts find their analogue in general von Neumann algebras and illustrate some of the complexities that arise. A von Neumann algebra is a subalgebra A of the algebra of all bounded operators acting on a Hilbert space H which is closed under the natural involution of taking adjoints and is equal to its own double commutant. Unlike the case of M_n, there are several topologies. For example, the norm topology and the σ-weak topology induced by the functionals $x \to \Sigma(x\alpha_i, \alpha_i)$ where $\{\alpha_n\}$ is a sequence in H with $\Sigma\|\alpha_1\|^2 < \infty$. A subalgebra of the algebra of bounded operators on H is a von Neumann algebra if and only if it contains the identity and is σ-weakly closed. The σ-weakly continuous functionals form a Banach space in the dual of A and A is its dual space. This gives an abstract characterization of a von Neumann algebra: a C^*-algebra (i.e. a Banach algebra with involution with the property $\|x\|^2 = \|x^*x\|$) that is the dual space of a necessarily unique Banach space called the predual. It turns out that C^*-algebras are just the class of norm closed *-subalgebras of the algebra of all bounded operators on some Hilbert space.

Two examples of von Neumann algebras appear: the infinite analogue of M_n equal to the algebra $L(H)$ of all bounded operators on a Hilbert space H and the algebra L^∞ of essentially bounded measurable functions on locally compact space with respect to a Radon measure. The latter is equal to the dual of L^1 and is the form of all abelian von Neumann algebras.

There are several ways to generate von Neumann algebras from others as taking direct products, tensor products, reducing to smaller Hilbert spaces. One of the fundamental questions is then to classify von Neumann algebras, preferably while doing so to illuminate as much as possible the structure of C^*-algebras. It is part of the work of J. von Neumann that a von Neumann algebra A acting on separable Hilbert space can be written as a certain continuous sum (called a direct integral) of factor von Neumann algebras (i.e., those with center equal to the scalar multipliers of 1) acting on a direct integral of separable Hilbert spaces. The direct integral theory is nontrivial in the sense that A is not isomorphic to a factor tensor a commutative algebra. It is presumed and actually verifiable in many cases that the properties of the original algebra will be reflected in the factors occurring in the direct integral of A. So one strategy has been to consider factors. A global theory, however, does exist.

From work dating back to Murray and von Neumann all factors are of one of the types I_n, I_∞, II_1, II_∞, III. The type I_n is just M_n $(n < \infty)$ and type I_∞ is $L(H)$ with dim $H = \infty$. The type II_1 has a trace and each projection p can be written as the sum of two projections p_1, p_2 with $p_1 \sim p_2$, and the type II_∞ is type $\mathrm{II}_1 \otimes L(H)$. Finally, type III is characterized by the fact that every projection p can be written as the sum of two projections p_1, p_2 with $p \sim p_1 \sim p_2$.

It is remarkable that some 15 years ago nonisomorphic examples of type II_1 and type III factors were being added one at a time. Finally, a continuum of both was found and then a powerful method for studying factors was developed.

For this let A be a von Neumann algebra on the Hilbert space H. Suppose there is a vector α that is cyclic (i.e., $A\alpha$ is dense in H) and separating for A (i.e., $x\alpha = 0$ implies $x = 0$ for x in A). The map $x\alpha \to x^*\alpha$ defines a preclosed densely defined linear operator on H. Its closure has the polar decomposition $J\Delta^{1/2}$ where J is an isometric involution of H and Δ is a positive selfadjoint operator called the modular operator. The functional calculus of unbounded operators shows that Δ^{it} is a unitary operator and it is a result of M. Tomita that $\sigma_t(x) = \Delta^{it}x\Delta^{-it}$ forms a one-parameter group of automorphisms of A. This group is called the modular automorphism group. Also it is true that $(\sigma_t(x)\alpha, \alpha) = (x\alpha, \alpha)$ for all x in A and real t, and that, for x, y in A, $t \to (\sigma_t(x)y\alpha, \alpha)$ and $t + i \to (\sigma_t(y)x\alpha, \alpha)$ form the boundary values of a continuous, analytic function defined on a closed strip of the complex plane.

Now, denoting Δ by Δ_α the intersection $S(A)$ of the spectra of all operators Δ_α (α cyclic and separating for A) turns out to be a closed subgroup of the reals (A. Connes, A von Daele) and since all such subgroups are known a natural invariant for distiguishing algebras is present. A. Connes showed that all possible subgroups occur in practice by forming a new crossed product algebra out of an algebra and a given automorphism group of the algebra. M. Takesaki then showed that every type III algebra is the crossed product of a type II_∞ algebra with a one-parameter automorphism group.

Now the problem of classifying von Neumann algebras reduces to that of classifying type II_1 algebras. Some important strides have already been made by A. Connes again by studying automorphism groups.

The main feature and value of the book *Lectures on von Neumann algebras* is the discussion in Chapters 9 and 10 of modular automorphisms, Hilbert algebras, modular Hilbert algebras and self-polar forms. These are developed for normal semi-finite weights in a manner that parallels the development for quasi-unitary algebras from traces. (The prototype for this is M_n acting on the Hilbert space M_n.) Included is a complete discussion of the theory of closed operators on Hilbert space that is necessary for the discussion of the modular operator. The discussion lacks a little of the clarity of earlier chapters in that much information is carried in general discussion paragraphs rather than in definitions or theorems. Yet the value of having the information all in one place with one set of notation is certainly very great.

The discussion concerning different types of von Neumann algebras is limited to the definition of the different types of algebras and the definition of their basic properties as found in other texts. Connes' classfication of factors is discussed briefly in the appendix of Chapter 10. In fact, two sections (a section *E* and *C*) are appended to each chapter: The first being exercises or additional theorems that the reader has a chance of doing for himself and the second being additional related topics that are briefly discussed.

There is no discussion of abelian von Neumann algebras or direct integrals.

Finally, the bibliography has over 120 pages.

Reports of new books on von Neumann algebras and C^*-algebras are circulating. In particular, books from R. V. Kadison and J. Ringrose, from M. Takesaki and from G. K. Pedersen are expected shortly.

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Principles of optimal control theory, by R. V. Gamkrelidze, Mathematical Concepts and Methods in Science and Engineering, Vol. 7, Plenum Press, New York, 1978, xii + 175 pp.

The present book is an account of lectures given at Tblissi State University. It is an excellent exposition of mathematical principles underlying Optimal Control Theory. In particular the author derives the maximum principle and establishes basic existence theorems for a special but typical optimal control problem. One of the outstanding features of the book is the clear presentation of the concept of relaxed or generalized controls which play such an important role in Optimal Control Theory.