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BOOK REVIEWS

Differential geometry, Lie groups and symmetric spaces, by Sigurdur Helgason, Academic Press, New York, 1978, xvi + 628 pp., $27.00.

This review will be divided into three parts:

1. What is (or should be) a Lie group?
2. Contents of Helgason's book.
3. Comments on Helgason's book.

1. A *Lie group* is roughly speaking a group parametrized by finitely many real parameters. Before worrying about a formal definition, let us look at some basic examples. The groups in these examples are not only illustrations of the Lie theory but are, in some sense, coextensive with it.

Let $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ denote the fields of real numbers, complex numbers, and the noncommutative field of real quaternions. The letter $\mathbf{F}$ will stand for any one of these fields. Let V be an n-dimensional vector space over $\mathbf{F}$. In case $\mathbf{F} = \mathbf{B}$, we take V to be a *right* vector space.

EXAMPLE 1. $\mathbf{R}^n$ with the usual additive structure.

EXAMPLE 2. $GL(V) =$ the group of all invertible linear transformations of V with multiplication given by composition. With a choice of a basis it may be identified with the group $GL_n(\mathbf{F})$ of $n \times n$ invertible matrices over $\mathbf{F}$.

$SL(V)$ is the subgroup of $GL(V)$ consisting of orientation preserving Lebesgue measure preserving transformations of V. In case $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, $SL(V)$ can be identified with the subgroup $SL_n(\mathbf{F})$ of $GL_n(\mathbf{F})$ consisting of matrices of determinant 1. In case $\mathbf{F} = \mathbf{H}$, we may regard V as a $2n$-dimensional vector space $V^{\mathbf{C}}$ over $\mathbf{C}$ by restriction of scalars. Then $GL(V) \subseteq GL(V^{\mathbf{C}})$ and $SL(V)$ can be realized as $GL(V) \cap SL(V^{\mathbf{C}})$.

EXAMPLE 3. Let Q be a nonsingular quadratic form on V which is either bilinear or sesquilinear (with respect to standard conjugations of $\mathbf{C}$ and $\mathbf{H}$), and either symmetric or skew. Let $O(Q)$ denote the subgroup of $GL(V)$ consisting of transformations preserving Q and $SO(Q) = O(Q) \cap SL(V)$. In detail, we then have the following seven families.

(a) $\mathbf{F} = \mathbf{R}$.

(i) A nonsingular symmetric bilinear form on V is characterized by its signature. If Q has signature h, then we shall write $O(p, q)$ for $O(Q)$, where $p + q = n, p - q = h$. Similarly, $SO(p, q)$ for $SO(Q)$.

(ii) A nonsingular skew bilinear form exists only if n is even, say $n = 2m$, in which case it is unique (up to isomorphism of quadratic spaces). One then has $O(Q) = SO(Q)$, which will also be denoted by $Sp_{2m}(\mathbf{R})$.

(b) $\mathbf{F} = \mathbf{C}$.

(i) Up to isomorphism V admits a unique nonsingular symmetric quadratic form. We write $O(n, \mathbf{C})$ for $O(Q)$ and $SO(n, \mathbf{C})$ for $SO(Q)$.

(ii) As in case (a), a nonsingular skew bilinear form exists only if n is even, say $n = 2m$, in which case it is unique up to isomorphism. Again, one has $O(Q) = SO(Q)$, which we now write as $Sp_{2m}(\mathbf{C})$.

(iii) A nonsingular symmetric sesquilinear form is characterized by its signature. If the signature of Q is $h, p + q = n$ and $p - q = h$, then we write $U(p, q)$ resp. $SU(p, q)$ for $O(Q)$ resp. $SO(Q)$. (If Q is a skew sesquilinear form, then iQ is symmetric sesquilinear and $O(Q) = O(iQ)$, so we get no new groups by considering skew sesquilinear forms.)

(c) $\mathbf{F} = \mathbf{H}$. In this case there are no bilinear quadratic forms.

(i) A nonsingular symmetric sesquilinear form Q is characterized by its signature, and $O(Q) = SO(Q)$. If the signature of Q is $h, p + q = n, p - q = h$, we write $Sp(p, q)$ for $O(Q)$.

(ii) Up to isomorphism there exists a unique nonsingular skew sesquilinear form Q on V and $O(Q) = SO(Q)$, which we also write as $Sp_n(\mathbf{H})$.

What is interesting about these groups? Although the answers to such a question are subjective, with hindsight one can offer some very precise answers to this question. I shall give four in addition to

(0) These groups are clearly interesting in themselves. In particular, the algebraic structure of these groups and the topological and geometric features of their standard actions on V or the associated projective spaces have led to many pretty researches. E.g., see [**1**].

(1) *Classical geometries.* The most classical and familiar geometry is the Euclidean geometry of the plane. The full group $\mathcal{E}(2)$ of Euclidean motions of $\mathbf{R}^2$ is not listed above but can be obtained as a semidirect product: $\mathcal{E}(2) \approx \mathbf{R}^2 \rtimes O(2)$, where $O(2)$ stands for $O(2, 0)$ and consists of rotations around the "origin", $\mathbf{R}^2$ is the normal subgroup of $\mathcal{E}(2)$ consisting of translations, and $O(2)$ acts on $\mathbf{R}^2$ in a standard way. Now let us ask, what *is* the plane Euclidean geometry? One's immediate reaction is, it is the study of subsets of $\mathbf{R}^2$ "up to congruence", which precisely means, *it is the study of those properties of subsets of* $\mathbf{R}^2$ *which are invariant under the action of* $\mathcal{E}(2)$. This is Klein's viewpoint. Instead of saying that 'a geometry gives rise to a group', Klein inverted the phrase: 'a geometry is *defined* by a group'. Another example: the usual geometry of the unit sphere S^2 is simply the geometry *defined* by the usual action of $O(3)$.

Here is a much more sophisticated example, which in conception goes back to Riemann, which inspired and thrilled the generation of Klein and Poincaré, and which retains its beauty even today. Consider the upper half plane $H = \{z = x + iy \in \mathbf{C} | y > 0\}$. The group $G \approx SL_2(\mathbf{R})/\pm 1$ of real Möbius transformations

$$z \to \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbf{R},$$

$ad - bc = 1$, acts transitively on H. This group may be regarded either as the full group of orientation-preserving isometries of H equipped with the hyperbolic metric $|dz|/y$ of constant negative curvature, or as the full group of holomorphic transformations of H considered as a Riemann surface. As a Riemann surface, the significance of H is that it is the universal covering surface of all Riemann surfaces except the sphere, the elliptic curves, and once or twice punctured sphere. It is thus the *common group* G which allows one to introduce the geometric ideas in the function theory of one complex variable and vice versa.

There are several higher-dimensional analogues of these examples. E.g., $\mathbf{R}^n \approx \mathfrak{E}(n)/O(n)$, where $\mathfrak{E}(n) \approx \mathbf{R}^n \rtimes O(n)$ is the group of Euclidean motions, $S^n \approx O(n+1)/O(n)$, $H^n \approx O_0(n, 1)/O_0(n)$[1] are respectively the model spaces of n-dimensional Euclidean, spherical and hyperbolic geometries. Then there are complex and quaternionic analogues of these examples. E.g., $\mathbf{P}_n(\mathbf{C}) \approx U(n+1)/U(n) \times U(1)$ is the space of complex projective geometry. Similarly, $U(p+q)/U(p) \times U(q)$ is the grassmannian of p-planes in $\mathbf{C}^{p+q}$ with a distinguished geometry. Along with these there are noncompact "hyperbolic" analogues such as $U(p, q)/U(p) \times U(q)$, which for $q = 1$ is the unit complex ball in $\mathbf{C}^p$ admitting a metric of constant negative holomorphic sectional curvature.

(2) *Differential geometry and topology*. Let M be an n-dimensional differentiable manifold. A *frame* at $p \in M$ is an ordered set of n linearly independent tangent vectors at p. Consider the set $F(M)$ of all frames at all points in M. Then $GL_n(\mathbf{R})$ acts on $F(M)$ via: if $\mathbf{f} = \{v_1, \ldots, v_n\}$ is a frame at $p \in M$, $A = (a_{ij}) \in GL_n(\mathbf{R})$, then $\mathbf{f} \cdot A = \{\sum_i v_i a_{ij}\}$. This action is clearly simply transitive on the set of all frames at each point. In modern terminology (with appropriate topology on $F(M)$) we have turned $F(M)$ into a principal fiber bundle with fiber $\approx GL_n(\mathbf{R})$. Now equip M with a Riemannian metric. Then one can talk of *orthonormal* frames, and clearly the set of orthonormal frames is a subbundle of $F(M)$ and is in itself a principal fiber bundle with fiber $O(n)$. *Conversely*, the choice of such a subbundle can be used to *define* a Riemannian metric (hence a "geometry") on M. This is Élie Cartan's viewpoint, also known as "the method of moving frames". While Klein views a "geometry" as defined on some distinguished space on which some distinguished (Lie) group acts transitively, Cartan views a distinguished Lie group already present at the local (even infinitesimal) level on an arbitrary space defining its 'geometry'. In this way Cartan put Lie groups at the very foundation of differential geometry.

The theory of tangent or normal bundles or various frame bundles on a differentiable manifold finds a natural generalization in the theory of fiber bundles on an arbitrary topological space, and the Lie groups continue to play their role in this theory.

To get some feeling for "moving frames" consider the Frenet Serret equations for a curve γ in $\mathbf{R}^3$ and muse *why* the equations involve a skew symmetric matrix. Interpreted properly, the skew symmetric matrix is just a tangent vector at identity on a certain curve associated to γ on $O(3)$!

(3) *Analysis*. Let (M, μ) be a measure space. Consider the problem of finding some "special functions" in $\mathfrak{L}^2 = \mathfrak{L}^2(M, \mu)$ such that any $f \in \mathfrak{L}^2$ is a "superposition" of these special functions.

EXAMPLES. (a) $M = S^1$ (the unit circle), or $\mathbf{R}$ with Lebesgue measure. The classical Fourier theory expresses $f \in \mathfrak{L}^2$ as a series resp. integral in the special functions $\{e^{in\theta} | n \in \mathbf{Z}\}$ resp. $\{e^{i\alpha x} | \alpha \in \mathbf{R}\}$ for $M = S^1$ resp. $\mathbf{R}$.

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[1]If G is a topological group, then G_0 stands for its identity component.

(b) $M = S^2$ (the unit sphere). Then $f \in \mathfrak{L}^2$ can be expressed as a series in spherical harmonics.

From many sources such special functions arise in mathematical physics as eigenfunctions of some distinguished differential operators, e.g., the Laplacian in the examples above. It was gradually realized that the spaces in question admit actions of large *Lie groups*, the differential operators in question are distinguished precisely by the invariance under these groups, and the special functions occur as matrix coefficients in unitary representations of these groups. In Example (a) above, the group is $SO(2)$ resp. $\mathbf{R}$ according as $M = S^1$ resp. $\mathbf{R}$; in Example (b), the group is $SO(3)$. Indeed, this is a deeper understanding of the phenomena. Nowadays in several sciences the knowledge of structural symmetries of an object combined with group theory is used to draw *a priori* conclusions, and this facilitates search for mathematical models.

(4) *Locally compact, connected, topological groups*. Granted that the concept of a group is a useful mathematical concept and the class of locally compact topological spaces is a reasonable class of spaces, one is naturally led to the question: what is the structure of a locally compact, connected, topological group? Group-theoretically, modulo the extension problem, the question is reduced to describing *simple groups*, i.e., ones without proper normal subgroups. Now the groups listed in the examples above have an obvious locally compact topology. Among these $SL(V)$'s are connected, and $SO(Q)$'s have at most two components. Let $SO_0(Q)$ denote the connected component of the identity. It is easily seen that except for some low-dimensional exceptions all these groups have finite center, which we denote by $Z(G)$, and, moreover, $G/Z(G)$ is a *simple* group (cf. [7]) with the only exceptions $SO(4)$, $SO(4, \mathbf{C})$, and $SO(2, 2)$, which modulo their centers are products of simple groups. One remarkable conclusion arising from the solution of Hilbert's fifth problem (cf. [3, p. 193]) is that *except for* 22 *exceptional groups these are all the connected*[2] *locally compact simple topological groups*. It should be remarked that some of the exceptional groups have been associated with geometries defined over Cayley numbers.

The above reasons for how, why, and where of the examples listed above are far from complete. We have not even mentioned the role of these groups in number theory, topological transformation groups, atomic physics, particle physics .... One may wonder and ask why do these groups occur in mathematics in so many ways? The question is metamathematical, so must be its answer. One may as well ask: why is the grass green and the sky blue? The answer is simply that it is in the nature of things. To elucidate: one of the most pertinent questions about any mathematical object is–what is the group of its structural symmetries? Now a "general" object may not have any symmetries at all. This is undoubtedly one of the reasons that while Lie tried to sell his work as the "Galois theory of differential equations", the people working in the general theory of differential equations are least bothered about Lie groups. On the other hand, an "interesting" object does exhibit

[2]The hypothesis of connectedness is essential. Without it one runs into other problems, an important one among which is the problem of determination of finite simple groups.

symmetries. An icosahedron, a finite Galois field extension, a regular branched or unbranched covering space have a discrete group of symmetries: the unit sphere, the hyperbolic plane, the Laplacian have a continuous group of symmetries. If the group is describable by *finitely many* real parameters, then it is a Lie group. One of the reasons which makes group theory profound is that quite different mathematical objects may have the same abstract group of structural symmetries, and then it is the common group which brings about unexpected unification.

2. We now come to the contents of Helgason's book. A part of the work has already been done. Helgason's book deals with items (1) and (2) of §1. Helgason develops differential geometry and the theory of Lie groups with the aim of classification of real semisimple Lie groups and symmetric spaces. By definition a *Lie group* is (1) a group, (2) a differentiable manifold, (3) the operations of group multiplication and inverse are differentiable. E.g., $GL_n(\mathbf{R})$ as an open subset of $\mathbf{R}^{n^2}$ is clearly a Lie group. Other groups considered in §1 are *closed* subgroups of $GL_n(\mathbf{R})$ for suitable n–in fact, they are zero sets of *polynomials* in the usual coordinates for $GL_n(\mathbf{R})$. Modulo some generalities of Lie theory and/or algebraic geometry, this is sufficient to ensure that all examples described in §1 are Lie groups. For the purpose of this review a *semisimple* Lie group may be defined as a Lie group G whose connected component of identity G_0 has discrete center $Z(G_0)$ so that $G_0/Z(G_0)$ is a product of simple groups (cf. §1, (4)). The underlying spaces of classical geometries (cf. §1, (1)) are all examples of *symmetric spaces*. By definition, a symmetric space is (1) a connected Riemannian manifold M, (2) $\forall p \in M\ \exists$ an isometry σ_p: $M \to M$, which fixes p and reverses the direction of each geodesic through p, in particular $\sigma_p^2 = 1$. A moment's reflection shows that condition (2) is a very stringent requirement from a space. E.g., it readily implies that the group G of isometries of M is transitive, so $M \approx G/K$. By some essentially topological generalities G is a Lie group and the isotropy group K is compact. It then follows essentially for group-theoretical reasons that there is a one-to-one correspondence among the following sets:

$$\begin{aligned} &\{\text{noncompact, connected simple Lie groups}\} \\ \overset{\phi}{\leftrightarrow}\ &\{\text{noncompact, irreducible symmetric spaces} \not\approx \mathbf{R}\} \\ \overset{\psi}{\leftrightarrow}\ &\{\text{compact irreducible simply connected symmetric spaces}\}. \end{aligned}$$

Here "irreducible" means that the space cannot be written as a product of Riemannian manifolds $M_1 \times M_2$, $\dim M_i > 0$, $i = 1, 2$. Each set contains 10 infinite families and 17 exceptions. In the first set we have already encountered the 10 families in §1. The correspondence ϕ goes as follows. If G is a noncompact connected simple Lie group, let K be its maximal compact subgroup which exists and is unique up to conjugation. Then $\phi(G) = G/K$ with a metric derived naturally from the so-called Killing form. Conversely, if M is a noncompact irreducible symmetric space $\not\approx \mathbf{R}$, then $\phi^{-1}(M) = G_0/Z(G_0)$, where G_0 is the identity component of the group of isometries of M. The correspondence ψ sets up the "duality" between the noncompact and compact symmetric spaces. Although it is not hard to describe for reasons of space we omit it here. Here is a significant example of this duality:

$$\begin{array}{ccc} S^n \approx O(n+1)/O(n) & \overset{\psi}{\leftrightarrow} & H^n \approx O(n,1)/O(n) \\ = \text{the spherical geometry} & & = \text{the hyperbolic geometry.} \end{array}$$

The correspondences ϕ and ψ are a beautiful discovery of Élie Cartan. He also made a differential-geometric refinement: a Riemannian manifold is *locally symmetric* (the definition is obvious) iff the curvature tensor is parallel. Moreover, a complete, *simply connected*, connected locally symmetric space is symmetric.

Among the symmetric spaces a particularly interesting class from the compact-analytic viewpoint is that of hermitian symmetric spaces. These are group-theoretic generalizations of the Riemann sphere and the Poincaré disc. The compact ones among these are some remarkable algebraic varieties whereas the noncompact ones are a natural ground for higher-dimensional generalizations of the classical theory of automorphic forms.

All this involves some very general topological and differential geometric principles and the rich internal structure of connected simple Lie groups. The main step then is the classification and elucidation of the structure of connected simple Lie groups. This is the pioneering work of Lie, Killing, Cartan, Weyl .... Its exposition occupies about four fifths of this book. In a review like this, the details of this work can be given only very sketchily.

Step 1 (*Linearization*). To each Lie group G (which, incidentally, was not rigorously defined either by Lie or by Cartan!), Lie associated what we now call a *Lie algebra* $L(G)$. E.g., the Lie algebra of $GL_n(\mathbf{R})$ is $M_n(\mathbf{R}) = n \times n$ real matrices with usual addition and the product (usually denoted by $[\ ,\ ]$) given by $[A, B] = A \circ B - B \circ A$, where $\circ$ denotes the usual matrix multiplication. As another example, the Lie algebra of $SO(n)$ can be thought of as consisting of $n \times n$ real skew symmetric matrices; for $n = 3$ this may be further identified with $\mathbf{R}^3$ with product given by the the usual cross product of vectors. The *main theorem of Lie theory* (in large part due to Lie) is the grand isomorphism of categories

$$\{\text{Connected simply connected Lie groups}\} \leftrightarrow \{\text{Real Lie algebras}\}.$$

The reverse map from a Lie algebra to a Lie group is called the *exponential map*. For Lie subgroups of $GL_n(\mathbf{R})$, it is given by the matrix exponentiation $A \rightsquigarrow e^A$.

Step 2 (*Group theoretical concepts*). In the correspondence above between Lie groups and Lie algebras, Lie subgroups resp. normal Lie subgroups correspond to subalgebras resp. ideals. This allows one to transfer the standard group theoretical concepts such as nilpotency, solvability, simplicity[3] ... of a Lie group to the corresponding notions for its Lie algebra.

Step 3 (*Linear algebraic concepts*). Due to Lie's theorem the classification problem for Lie groups (modulo the covering space theory) reduces to that for Lie algebras. The latter is accessible to linear algebraic techniques. Let L be a Lie algebra, V a vector space both over either $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, and ρ:

[3]In the case of Lie groups, one usually defines these notions only for the identity component and up to factoring by discrete central subgroups.

$L \to \operatorname{End} V$ a representation of L on V over $\mathbf{F}$. If $v_0 \in V$ is a *common* eigenvector for $\rho(x)$, $x \in L$, then the corresponding eigenvalues give rise to a homomorphism λ: $L \to \mathbf{F}$. Such a homomorphism is called a *weight* and $V_\lambda = \{v \in V | (f(x) - \lambda(x))^{\dim V} v_0 = 0\}$ is called the corresponding *weight space*. Generalizing the usual Jordan decomposition of an endomorphism of a vector space one gets: if $\mathbf{F} = \mathbf{C}$ and L is a *nilpotent* Lie algebra, then $V = \bigoplus V_\lambda$, where λ ranges over weights.

An important canonical representation for any Lie algebra L is the adjoint (or left regular) representation ad: $L \to \operatorname{End} L$ given by $(\operatorname{ad} x)(y) = [x, y]$. (If $L = L(G)$, then ad is associated to the canonical action of G on itself by conjugation.) Now the weights of suitable *nilpotent* subalgebras of L in the adjoint representation are precisely the invariants which contain a good deal of structural information and which in paricular lead to the classification of simple algebras. It was this way that Killing [5] made his remarkable discovery that *besides the Lie algebras of* (1) $SL_n(\mathbf{C})$, $n \geqslant 2$, (2) $SO(n, \mathbf{C})$, $n \geqslant 3$, $n \neq 4$, *and* (3) $Sp_{2n}(\mathbf{C})$, $n \geqslant 1$, *there are precisely five exceptional complex simple Lie algebras*.

Finally, if L is a real Lie algebra, then $L \otimes_{\mathbf{R}} \mathbf{C} = L_{\mathbf{C}}$ is a complex Lie algebra and L is called a *real form* of $L_{\mathbf{C}}$. The classification of real simple Lie algebras then easily reduces to finding real forms of complex simple algebras. This very complicated problem was solved by Élie Cartan. The final result: *besides the Lie algebras of simple Lie groups among the groups listed in* §1 *and the five exceptional complex simple Lie algebras considered as real Lie algebras by restricting scalars there are precisely* 17 *exceptional real simple Lie algebras all of which are real forms of the* 5 *exceptional complex simple Lie algebras mentioned above*.

3. By now the reader should get some feeling that presenting this magnificent theme in sufficient detail is a gigantic task. Helgason has accomplished it–and in a very competent way. This book, published in 1978, is a thoroughly revised and updated version of the author's well-known book published in 1962 under the title *Differential geometry and symmetric spaces*. The reader may note that the words "*Lie groups*" have been added in the title of the present book. The number of pages has gone up from 486 to 628–and this is only the first volume. The author promises a sequel which will deal with function theory on symmetric spaces. The chapter on function theory in the '62 book has been dropped and a new chapter on the structure of semisimple Lie groups has been added. The '62 book dealt with the classification of semisimple Lie groups and symmetric spaces rather sketchily. The new book contains more details along the ideas in Kač-Moody algebras (cf. [4], [6]), which appeared in 1968 and which have significantly clarified and simplified the combinatorial aspects of the theory. Helgason has made a conscientious effort to make the book accessible to a wider public. A rather unusual feature for a book of this type is some 50 pages devoted to the solutions of the exercises in the book. All these are very welcome additions. The '62 book has served as a standard reference book for the last 16 years. It may be safely predicted that the new book will continue to do so. Many of Lie's and Cartan's intuitive assertions were given a rigorous treatment in the first edition of this book. Despite the appearance of many texts and notes during the past two decades, Helgason's book with the new additions remains the only complete and exhaustive reference on symmetric spaces.

Some critical comments and suggestions are in order. It appears that the '62 book was admittedly a reference book. Helgason suggests that the new book is also a *textbook* and may be used for student seminars. The reviewer has reservations on this point. The book is technically self-contained. But due to its very scope and the kaleidoscopic view of the field, beginning students would find the presentation quite overwhelming in many places, and would easily get lost without very substantial and competent help from an instructor. Moreover the book is mercilessly demanding on the reader's constant vigilance. In part this is due to the take-off points Helgason has chosen. Symmetric spaces are *primarily* homogeneous spaces which happen to have a geometrically intuitive characterization. Undoubtedly what makes the theory of symmetric spaces tick is the rich internal structure of semisimple Lie groups. Helgason's take-off point is the much less intuitive condition $\nabla R = 0$, which happens to characterize a locally symmetric space. Then he fills in by piecemeal the structure theory of semisimple Lie groups. Another example: for Helgason the exponential map from a Lie algebra to a Lie group is a special case of the exponential map of a manifold with affine connection. This is very pleasing for an experienced differential geometer, but for most others (probably including Lie!) it would be a rather painful experience to decipher its meaning. About seven terse pages after the definition of the exponential mapping the reader is told that for $GL_n(\mathbf{R})$ it is the same as the matrix exponentiation. (Actually one could even start with the matrix exponentiation as a *definition* of the exponential mapping for Lie subgroups of $GL_n(\mathbf{R})$ and then modulo Ado's theorem extend it to a general Lie group by the standard covering space theory.)

There is also a matter of the choice of terminology. Must one say 'a compact symmetric space of type A III in Cartan's list' instead of just 'a complex Grassmannian'? For the same reason the reviewer sees little point in the notation $SU^*(2n)$ for $SL_n(\mathbf{H})$ and $SO^*(2n)$ for $Sp_n(\mathbf{H})$. Incidentally, the connection of $SU^*(2n)$ and $SO^*(2n)$ to quaternions is not pointed out by Helgason.

On the differential geometric side it would be of interest to include the discussion on holonomy in Chapter 1 especially since the symmetric spaces provide crucial illustrations of these general concepts. (After its brief appearance on p. 197, the holonomy group appears on p. 427 without warning in a rather crucial way.)

On the topological side the role of the fundamental group and simple connectivity should be clearly pinpointed very early in the development somewhere. This will uniformly bring more clarity later, e.g., in the discussion of contrasts between the noncompact and compact symmetric spaces. It is somewhat surprising that there is nowhere an explicit statement about the one-to-one correspondence between real Lie algebras and *simply connected* connected Lie groups although, of course, it is used at many places.

As noted earlier, because of its piecemeal development the group theory has somewhat suffered. Ado's theorem is omitted. The Levi decomposition is barely mentioned on page 147 but doesn't even appear in the index. The definition of a Cartan algebra is quite unmotivated and is valid only in the semisimple case. The Euclidean case is so "trivial" that the Euclidean group of motions receives no comment anywhere. The interrelations between the notions of solvability, simplicity, etc. of Lie algebras and the corresponding notions for groups are not clarified. Such omissions may be acceptable for someone with some experience in the field but they certainly make the going rough for a newcomer.

I must hasten to add, however, that the mathematical community owes a great debt to Helgason for making this beautiful subject accessible in a book form with competence and with infectious enthusiasm. To be sure, the Lie theory has made more demands on Helgason than on us, who would be benefitting by reading his book. The reader should not miss the introduction, the notes, and the descriptive passages in each chapter. Also, from the beginning the reader should keep in touch with Chapter 10, which discusses the examples. That is the meat of this subject. Then the reader would better savor the gravy of the theory, which has made that meat digestible.

Errata (contributed by the author)

In Theorem 3.29 p. 478 $\mathfrak{b}_1$ should be $\mathfrak{a}_1$, $\mathfrak{b}_1$ $\mathfrak{a}_1$ should be $\mathfrak{a}_1 (1 \geqslant 2)$

On p. 515, Table III $\mathfrak{a}_n$ $(n \geqslant 1)$ should be $\mathfrak{a}_n$ $(n \geqslant 1)$

On p. 534 line 13^- $(i = 2, 4)$ should be $(i = 6, 4)$

On p. 507 line 1^+ $\mathfrak{a}_{2n-1}$ should be $\mathfrak{a}_{2n-1}$ $(n > 2)$

On p. 151 line 4^- Theorem 6.9 should be Proposition 6.6(i)

Page 503 In $e_6^{(2)}$ the labels should be 1 2 3 2 1 not 1 2 3 1 2

On p. 458 line 11^- Theorem 2.12 should be Theorem 2.12 and Corollary 7

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