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ON NONVANISHING OF L-FUNCTIONS

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The nonvanishing of Hecke L-functions at the line $\mathrm{Re}(s) = 1$ has proved to be useful in the theory of uniform distribution of primes. One of the generalizations of this fact is due to H. Jacquet and J. A. Shalika [4], who proved the nonvanishing of the L-functions considered in [2]. The following theorem generalizes this result to the L-functions attached to the pairs of cusp forms on $GL_n \times GL_m$ (cf. [3]). It appears to have an application in the classification of automorphic forms on GL_n (communications with H. Jacquet and J. A. Shalika).

Let F be a number field and denote by $\mathbf{A}$ its ring of adeles. Fix two positive integers m and n. Let π and π' be two cuspidal representations of $GL_n(\mathbf{A})$ and $GL_m(\mathbf{A})$. Fix a complex number s. Write $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi'_v$, where π_v and π'_v denote the vth components of π and π' at each place v of F, respectively. Let S be the finite set of all ramified places, including the infinite ones. For every finite place v, H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika have defined (cf. [3]) a local L-function $L(s, \pi_v \times \pi'_v)$. Let

$$L_S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi'_v).$$

Put $i = (-1)^{1/2}$. Then we have

THEOREM. $L_S(1 + it, \pi \times \pi') \neq 0$ *for* $\forall t \in \mathbf{R}$.

OUTLINE OF THE PROOF. The proof follows the general principle of applying Eisenstein series to L-functions which is due to R. P. Langlands [5] (same as in [4]). Put $G = GL_{n+m}$ and $M = GL_n \times GL_m$. Consider M as a Levi factor of a maximal standard parabolic subgroup of G. Choose φ in the space of ${}^\circ\pi = \widetilde{\pi} \otimes \pi'$, where $\widetilde{\pi}$ denotes the contragredient of π. Extend φ to $\widetilde{\varphi}$, a function on $G(\mathbf{A})$, as in [7]. Put

$$\Phi_s(g) = \delta_P^{s-1/2}(p)\widetilde{\varphi}(g),$$

where $P = MN$, $g = kp$, $p \in P(\mathbf{A})$, and $k \in K$. Here $K = \Pi_v K_v$ is a maximal compact subgroup of $G(\mathbf{A})$ such that $K_v = G(O_v)$ for every finite v. Now set (cf. [6], [7])

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$$E(s, \tilde{\varphi}, g, P) = \sum_{\gamma \in G(F)/P(F)} \Phi_s(g\gamma),$$

the Eisenstein series attached to φ. Consider

$$E_\chi(s, \tilde{\varphi}, g, P) = \int_{U(A)/U(F)} E(s, \tilde{\varphi}, gu, P)\overline{\chi(u)}\,du,$$

where U is the subgroup of upper triangulars in G with ones on diagonals, and χ is a nondegenerate character of $U(\mathrm{A})/U(F)$. Now for each place v, let

$$\Pi_v = \operatorname*{Ind}_{P(F_v)\uparrow G(F_v)} ((^\circ \pi_v)_\infty \otimes \delta^s_{P,v})$$

and denote by λ_v the Whittaker functional attached to Π_v as in [1], [7], and [8]. Put

$$W_{s,v}(g) = \lambda_v(\Pi_v(g^{-1})f_{s,v}) \qquad (g \in G(F_v)),$$

where $f_{s,v}$ is defined as in Lemma 4.1 of [7]. Then for $\mathrm{Re}(s) < -1/2$,

$$E_\chi(s, \tilde{\varphi}, g, P) = \prod_v W_{s,v}(g_v) \qquad (g = (g_v) \in G(\mathrm{A})).$$

It is proved in [1] and [8] (also see [7]) that at every v, $W_{s,v}$ may be so chosen that $W_{s,v}(e) \neq 0$. Now write

$$E_\chi(s, \tilde{\varphi}, e, P) = \prod_{v \in S} W_{s,v}(e) \cdot \prod_{v \notin S} W_{s,v}(e).$$

Then by the previous remark, we may choose φ such that $\Pi_{v\in S} W_{s,v}(e)$ is nonzero.

It is a result of W. Casselman and J. A. Shalika [1] that if v is unramified, φ can be so chosen that

$$W_{s,v}(e) = L(-(n+m)s + 1, \pi_v \times \pi'_v)^{-1},$$

and therefore

$$L_S(-(n+m)s + 1, \pi \times \pi')^{-1} = \prod_{v \notin S} W_{s,v}(e).$$

Now the theorem follows from the fact that $E(it, \tilde{\varphi}, e, P)$ and consequently $E_\chi(it, \tilde{\varphi}, e, P)$ are both holomorphic for all $t \in \mathbf{R}$ (cf. [6]).

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