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REALIZATIONS OF NONLINEAR SYSTEMS AND ABSTRACT TRANSITIVE LIE ALGEBRAS

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Introduction. We shall derive nonlinear analytic realizations in system theory from infinite transitive Lie algebras and Lie pseudogroups. By using noncommutative generating power series, this new approach allows a local input-output viewpoint which was not possible until now (cf. [5], [6], [9], [10]) and should lead, thanks to the notion of *syntactic* Lie algebra, to many developments.

I. Review of noncommutative generating power series (cf. [2], [3]). X^* is the free monoid generated by $X = \{x_0, x_1, \ldots, x_n\}$ and 1 is its identity element. Let $\mathbf{R}\langle X\rangle$ and $\mathbf{R}\langle\langle X\rangle\rangle$ be the $\mathbf{R}$-algebras of formal polynomials and power series with real coefficients and associative variables $x_j \in X$ (noncommutative if $n \geqslant 1$).

A causal functional $F(t; u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$: $[0, T) \longrightarrow \mathbf{R}$ are piecewise continuous functions, is said to be *analytic* iff it is given by an element

$$g = (g, 1) + \sum_{\nu \geqslant 0} \sum_{j_0, \ldots, j_\nu = 0}^{n} (g, x_{j_\nu} \ldots x_{j_0}) x_{j_\nu} \ldots x_{j_0}$$

of $\mathbf{R}\langle\langle X\rangle\rangle$, called its *generating* power series, such that its value is

$$F(t; u_i) = (g, 1) + \sum_{\nu \geqslant 0} \sum_{j_0, \ldots, j_\nu = 0}^{n} (g, x_{j_\nu} \ldots x_{j_0}) \int_0^t d\xi_{j_\nu} \ldots d\xi_{j_0}. \tag{1}$$

The iterated integral is defined recursively on the length

$$\xi_0(\tau) = \tau, \qquad \xi_i(\tau) = \int_0^\tau u_i(\sigma)\, d\sigma \qquad (i = 1, \ldots, n),$$

$$\int_0^\tau d\xi_j = \xi_j(\tau) \qquad (j = 0, 1, \ldots, n),$$

$$\int_0^t d\xi_{j_\nu} \ldots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau) \int_0^\tau d\xi_{j_{\nu-1}} \ldots d\xi_{j_0}.$$

HYPOTHESIS (H). (1) is absolutely convergent for t and $\max_{0 \leqslant \tau \leqslant t} |u_i(\tau)|$ sufficiently small.

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Let us introduce the differential system

$$\begin{cases} \dot{q}(t)\ (= dq/dt) = A_0(q) + \sum_{i=1}^{n} u_i(t)A_i(q) \\ y(t) = h(q), \end{cases}$$

where $q(t)$ belongs to a real analytic manifold Q and $q(0)$ is given; $A_0, A_1, \ldots, A_n$ are analytic vector fields and $h\colon Q \longrightarrow \mathbf{R}$ is an analytic function, which are defined in a neighbourhood of $q(0)$. The output y is an analytic functional with the following generating power series

$$\mathfrak{g} = h|_{q(0)} + \sum_{\nu \geqslant 0} \sum_{j_0, \ldots, j_\nu = 0}^{n} A_{j_0} \ldots A_{j_\nu} h|_{q(0)} x_{j_\nu} \ldots x_{j_0},$$

where the bar $|_{q(0)}$ indicates the evaluation at $q(0)$.

II. Results. Let $\mathcal{H}(\mathfrak{g})$ be the Hankel matrix (cf. [1]) of $\mathfrak{g} \in \mathbf{R}\langle\langle X \rangle\rangle$. The $\mathbf{R}$-vector space spanned by its columns has the canonical structure of a left $\mathbf{R}\langle X \rangle$-module: the product $u \cdot c_v$ of $u \in X^*$ and the column with index $v \in X^*$ is the column $c_{uv} \in \mathcal{H}(\mathfrak{g})$.

$L(X)$ is the free Lie algebra the envelopping algebra of which is $\mathbf{R}\langle X \rangle$. The *Lie rank* of $\mathfrak{g}$ is the dimension of the $\mathbf{R}$-vector space spanned by $\{pc_1 | p \in L(X)\}$.

THEOREM. (A) *For* $\mathfrak{g} \in \mathbf{R}\langle\langle X \rangle\rangle$, *the two following conditions are equivalent*:

(i) *The Lie rank of* $\mathfrak{g}$ *is finite, equal to N.*

(ii) *There exist a commutative power series* $h \in \mathbf{R}[[q^1, \ldots, q^N]]$ *and formal vector fields*

$$A_j = \sum_{k=1}^{N} \theta_j^k \frac{\partial}{\partial q^k} \qquad (\theta_j^k \in \mathbf{R}[[q^1, \ldots, q^N]]\,; j = 0, 1, \ldots, n),$$

defined up to equivalence, such that

$$\mathfrak{g} = h|_0 + \sum_{\nu \geqslant 0} \sum_{j_0, \ldots, j_\nu = 0}^{n} A_{j_0} \ldots A_{j_\nu} h|_0 x_{j_\nu} \ldots x_{j_0},$$

where the bar $|_0$ *indicates the evaluation at* $q^1 = \cdots = q^N = 0$.

(B) *When hypothesis* (H) *is satisfied, it is possible to choose the series* θ_j^k *and h to be convergent in a neighbourhood of* 0. *This choice is unique up to equivalence.*

III. Sketch of the proof. Let $\mathcal{I}$ be the ideal of $L(X)$ defined by

$$\mathcal{I} = \{p \in L(X) | \forall k \geqslant 0, \forall p_1, \ldots, p_k \in L(X), [p_k, \ldots [p_1, p] \ldots] c_1 = 0\}.$$

We shall say that $\mathfrak{A}(\mathfrak{g}) = L(X)/\mathfrak{I}$ is the *syntactic* Lie algebra of $\mathfrak{g}$ which can be looked upon as an *abstract transitive* Lie algebra (cf. [4], [8]) with the following *fundamental* subalgebra:

$$L^0 = \{p \in \mathfrak{A}(\mathfrak{g}) \mid pc_1 = 0\}.$$

The Lie rank of $\mathfrak{g}$ is equal to the dimension of $\mathfrak{A}(\mathfrak{g})/L^0$. As a result, part A of the theorem follows from [4, Theorem III], or [8, Theorem 4.3].

When hypothesis (H) is satisfied, the corresponding local Lie pseudogroup can be constructed with the same kind of methods used in [6, §5]. The uniqueness up to equivalence follows from [8, §3.2], or [7].

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