BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0024/$02.00

Multiple-conclusion logic, by D. J. Shoesmith and T. J. Smiley, Cambridge Univ. Press, Cambridge, 1978, xiii + 396 pp., $35.00.

The subject of this book is meta-meta-mathematics; it is related to meta-mathematics as the latter is to the rest of mathematics. The aim of mathematics is to describe interesting objects and phenomena in precise terms, expressing their basic properties as axioms, and to deduce interesting theorems from these axioms. Meta-mathematics is the part of mathematics where the phenomenon to be described is the process of mathematical deduction. Its basic concept is the relation of logical consequence, which holds between a set X of statements and a statement A (written $X \vdash A$ by Shoesmith and Smiley although $X \vDash A$ is more common) when A is true in every conceivable situation where all the members of X are true. Its results, such as Gödel's completeness theorem, apply to all the various axiom systems of mathematics regardless of their meaning or purpose.

The definition of $\vdash$ is vague in that we have not specified what situations are conceivable. Traditionally, one regards the meanings of the logical connectives (not, and, or, . . .), the quantifiers (for all, for some), and equality as fixed, so that a situation where one of these has a nonstandard interpretation is considered inconceivable, but the meanings of other expressions are permitted to vary. The resulting $\vdash$ is the consequence relation of the classical first-order predicate calculus. If, in addition, we fix the meaning of "natural number", we obtain (the consequence relation of) a stronger logical system called ω-logic. If we fix the meaning of "set" we obtain second-order logic (which is stronger than ω-logic since the Peano axioms provide a characterization of the natural numbers). In the other direction, if we unfix the meanings of the quantifiers and equality, we obtain propositional logic.

Other consequence relations are produced by varying the meaning of "true". The most important of these are the constructive logics such as intuitionism, where "true" is taken to be synonymous with "proved" or with "provable". There are also many-valued propositional logics, where one has a set of two or more (usually more) truth values, some of which are designated as "true", and for each logical connective one has a corresponding operation on truth values. A simple but useful example has as truth values the four ordered pairs of ordinary truth values (true and false), with only $\langle$true, true$\rangle$ designated, and with the connectives operating componentwise. This example is rather special in that is has the same consequence relation as classical propositional logic.

Shoesmith and Smiley develop a theory of consequence relatons in general, intended to be applicable to all the preceding examples (and others). They work, however, with consequence relations, $\vdash$, both of whose arguments, not just the left one, are sets of statements. If $X \vdash Y$ were interpreted in the obvious way as "Any conceivable situation making all the statements in X true also makes all the statements in Y true", this would reduce to the single-conclusion concept of $\vdash$ since it is equivalent to "$X \vdash A$ for all $A \in Y$". In order to get an interplay between the elements of Y similar to what happens in X (where all the elements of X can jointly imply things that no one of them could imply), it is necessary to define $X \vdash Y$ as "Any conceivable situation making all the elements of X true also makes *at least one* element of Y true". Note that the multiple-conclusion consequence relation, unlike the single conclusion one, distinguishes between classical propositional logic and its four-valued cartesian square described above, since A or $B \vdash A, B$ (where the set-formation braces have been omitted in accordance with tradition) is correct in the former but not in the latter. (This is an instance of a general phenomenon explored in Chapter 17.)

The framework on which Shoesmith and Smiley construct their theory is this. There is a set V whose members are called formulas. A situation is a partition of V into a set T of true formulas and a set U of untrue ones. A logical system is a family of situations, considered conceivable, and the consequence relation $\vdash$ of such a system is defined as in the preceding paragraph. Of course the single-conclusion consequence relation can be extracted from the multiple-conclusion one by considering singletons on the right of $\vdash$.

Part I of *Multiple-conclusion logic* (Chapters 1–6) is about general properties of consequence relations. It contains, among other things, a direct characterization of consequence relations, a discussion of compact consequence relations (those for which $X \vdash Y$ only if $X' \vdash Y'$ for some finite subsets X', Y' of X, Y), and a discussion of the (usually large) collection of multiple-conclusion consequence relations with a specified single-conclusion part.

Part II (Chapters 7–12) is about proofs. The question considered here is: Given that a consequence relation $\vdash$ holds between certain pairs $\langle X_i, Y_i \rangle$, how can one prove that it also holds between $\langle X, Y \rangle$? One expects a proof to be some arrangement of formulas showing how to get from X to Y in a number of steps, each step being from some X_i to the corresponding Y_i. Some caution is needed, however, to prevent the juxtaposition of the correct (in classical logic) inferences (A or B) $\vdash A, B$ and $A, B \vdash$ (A and B) to yield the incorrect A or $B \vdash A$ and B. The authors depict arrangements of steps using graphs. They represent steps by horizontal strokes and formulas by circles (except that the elements of X are drawn as ∇ and those of Y as Δ), and they draw a line to (resp. from) each stroke from (resp. to) each premise (resp. conclusion) of the corresponding step. Thus, the incorrect juxtaposition above would be drawn as in Figure 1.

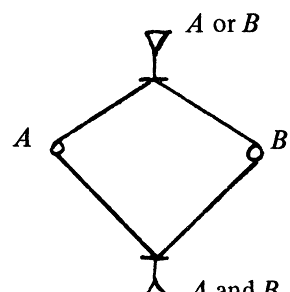

FIGURE 1

(Undirected lines are assumed to be directed downward.) A moment's reflection will lead the reader to suspect that the problem with this argument is the presence of a circuit in the graph. (What are commonly called circular arguments involve directed circuits; we see here that undirected circuits can also mean trouble.) And Theorem 8.1 confirms this suspicion by asserting that for an argument to be valid it suffices that it have no circuits and that it be "standard", i.e. that every formula occurring in the argument but not in X (resp. Y) be a conclusion (resp. premise) of some step in the proof. Yet, surprisingly, this sufficient condition is nowhere near necessary. The invalid argument above becomes valid when an appropriate directed circuit is added (as in Figure 2) or when several copies of it are appropriately joined together (see Figure 3).

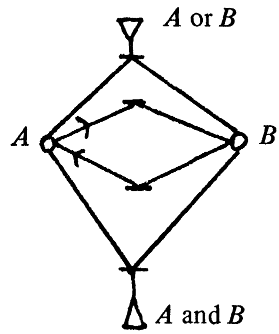

FIGURE 2

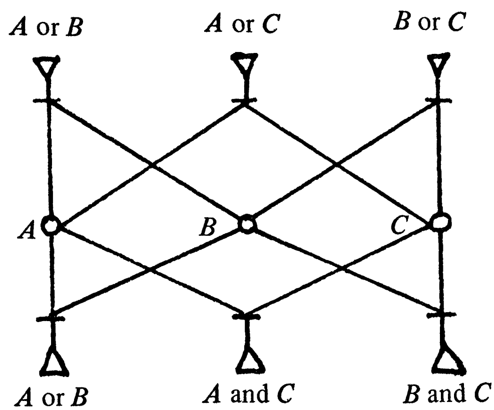

FIGURE 3

Even more surprisingly, there are correct inferences that cannot be obtained from circuit-free arguments (Theorem 8.10). Most of Part II is concerned with classes of arguments that are small enough to contain only valid arguments but broad enough to contain a proof of $X \vdash Y$ from any hypotheses that actually imply it. Theorems 10.3 and 10.4 describe one such class, the class of standard arguments in which every (undirected) circuit contains a "corner", i.e., a formula A whose two incident edges in the circuit are both directed toward A or both directed away from A. For example, the result proved in Figure 2 (one of whose circuits has no corner) is also proved by the cornered-circuit argument (Figure 4) from which Figure 2 can be obtained by identifying the two occurrences of A.

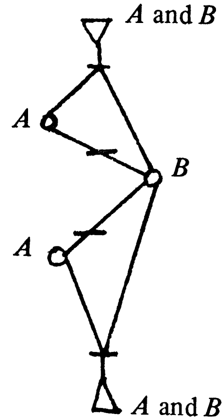

FIGURE 4

Note, incidentally, that the question whether a given graph argument for $X \vdash Y$ is valid is equivalent to whether a certain formula of propositional logic (in conjunctive normal form) is inconsistent; the formula contains as conjuncts (a) all the members of X, (b) the negations of all the members of Y, and (c) for each inference $X_i \vdash Y_i$ used in the argument, the disjunction of all the members of Y_i and the negations of all the members of X_i. Several of the results in the book are more intuitive when viewed in this way; for example, Theorem 7.6 reduces to the truth-table method of testing consistency.

Part III (Chapters 13–19) is concerned with systems of many-valued logic. Typical theorems are 15.2, which gives a criterion for a compact consequence relation to come from a many-valued system, and 17.11 and 17.12, which together say that an anti-well-ordered set of truth values, with the connectives given by its pseudo-Boolean algebra structure and with only the highest element designated, is uniquely characterized by the consequence relation it defines if and only if it is countable. (The importance of countability here is artificial, being traceable to the assumption that there are only countably many propositional variables, but the situation is in marked contrast to Theorem 17.16 which says that almost no many-valued systems are characterized by their single-conclusion consequence relations.)

Finally, Part IV (Chapter 20) is an attempt to develop natural deduction systems for the classical and intuitionistic predicate calculi in the framework of multiple-conclusion logic. For classical logic things work smoothly, but the intuitionistic case requires the introduction of a rather unnatural notion of "sound proof". Perhaps, however, this is to be expected since the partitions of V into the true and untrue statements in various situations, which are fundamental to the approach developed here, are highly discordant with the intuitionistic view of mathematics.

Although this book is about mathematics, it may well be more at home on the bookshelves of philosophers than of mathematicians. The subject matter is motivated by philosophical considerations, especially in the first two parts of the book. Applications are drawn primarily from many-valued logic, an area traditionally (though perhaps unjustly) linked to philosophy. Trains of thought that would naturally occur to a mathematician (or, at least, did occur to me) are omitted entirely. For example, there is no mention of the fact that, in discussing graph arguments from X to Y, one loses no generality by assuming that X and Y are both empty. ($X \vdash Y$ follows from a set of inferences $X_i \vdash Y_i$ if and only if $\varnothing \vdash \varnothing$ follows from these inferences along with $\varnothing \vdash A$ (resp. $A \vdash \varnothing$) for every formula A in X (resp. Y); in other words, ∇ and Δ can be replaced with $\overset{\top}{\circ}$ and $\underset{\bot}{\circ}$ respectively.) And, finally, the detailed motivations of concepts and theorems go well beyond the norm to which mathematicians are (unfortunately) accustomed.

ANDREAS BLASS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0025/$02.75

Invariant theory, by T. A. Springer, Lecture Notes in Math., vol. 585, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 111 pp., $8.00.

Invariant theory. The very words recall potent historical forces. Hilbert **[10]** viewed mathematical theory as the sum of three stages of development: the naive, the formal, and the critical periods. The progenitors of these periods come to mind. The Naive Period of invariant theory is represented by Boole, Sylvester, and Cayley, those conjurers of catalectants and other invariants of special quantics. The Formal Period arrived with the work of the Italian school of Cremona, Beltrami, and Capelli and the German school of Aronhold, Clebsch, and Gordon, whose symbolic method exposed the power of duality in algebra. In the Critical Period the heretic Hilbert reigned, armed with his homological methods; ultimately Noether, Van der Waerden and Artin enlarged on his ideas to found modern algebra.

Although Hilbert declared the subject dead in 1893 **[9]**, rumors of its demise were greatly exaggerated. Soon after, Reverend Young, alone and unnoticed, was divining the secrets of the symmetric group from his diagrams. At the same time, Weitzenböck, Study, and Littlewood unmasked tensor analysis as invariant theory in disguise. Soon after Molien, Frobenius, Cartan, Schur, and Weyl in generalizing invariant theory, ensconced it within a new subject, representation theory. No wonder Dieudonné could jest, "Invariant Theory has already been pronounced dead several times and like the phoenix it has been again and again arising from its ashes," **[3]**.

Has it been laid to rest? Hardly! The recent International Congress of Mathematicians in Helsinki included at least three forty-five minutes addresses devoted largely to recent progress in the field. Invariant theory is like the roots of a great tree, whose branches touch all of mathematics; still in its prime, it is bearing beautiful fruit. Consider some applications of invariant