as has been done for the classical three-body problem. Maybe it has been done somewhere.

The author is on less firm ground in the other sections on applications but it is still good reading. However, a statement as on the top of p. 153 "Our remarks apply with only obvious simple changes" (from the integers to the real line) shows some naiveté and includes three words (only, obvious, simple) which every mathematician knows may be cause for grave concern. On the other hand Mackey is clearly aware (in this situation) of the difficulties involved when natural orderings are not apparent. Perhaps it should be noted that the references given here to the probability literature are very incomplete.

To summarize, this is an extremely good book, written by a mathematician who is also a scientist and who is willing to make subjective statements to keep the theory alive and growing. It fills the bill in our current battles to revive the philosophy of mathematics as a part of a general scientific consciousness. It even passes the additional test of stating clearly certain open questions which remain in the theory and in the larger scientific investigations on which the theory may bear.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0020/$01.75

Nonlinear mappings of monotone type, by Dan Pascali and Silviu Sburlan, Sythoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1978, x + 342 pp., $43.00.

In the study of nonlinear problems much use is made of compactness arguments. Particularly since the work of Leray and Schauder [5], the compact operators have been widely used in this study and new applications continue to be made. However, in many interesting problems, the operators that arise fail to be compact. This has led, over the past two decades, to the search for classes of mappings, which actually arise in applications, for which nontrivial theorems can be proved. Several such classes have been studied (see e.g. [7], [**8**]), but most attention has been paid to the monotone operators. One good reason for this popularity is their appearance in many applications particularly in the theory of partial differential equations.

On the real line **R**, monotone operators are the monotone nondecreasing functions. In a Hilbert space H with inner product $(\cdot,\cdot)$, T: $H \to H$ is monotone if $(x - y, T(x) - T(y)) \geqslant 0$ for all x, y in H. Thus linear monotone operators in H are the well-known positive operators. To make the above definition work in a Banach space context, it is supposed that T map a Banach space X into its dual space X^* and $(\cdot,\cdot)$ then denotes the duality between X and X^*.

Consider the problem of minimizing a convex function ϕ: $H \to \mathbf{R}$. If ϕ is differentiable with gradient ϕ': $H \to H$ the problem of finding $\inf_{u\in H} \phi(u)$ becomes that of solving the equation $\phi'(u) = 0$. What property of ϕ' corresponds to convexity of ϕ? ϕ is convex if and only if ϕ' is monotone.

The study of the gradients of convex functions was replaced by the study of the larger class of monotone operators. The following result is probably the basic one.

Let T: $H \to H$ be a continuous, monotone operator such that $(x, T(x))/\|x\| \to \infty$ as $\|x\| \to \infty$. Then T maps H onto H, that is, the equation $T(x) = f$ has at least one solution for every f in H.

In fact, continuity is too strong; a weaker notion called hemicontinuity suffices. Also H can be replaced by a reflexive Banach space. One method of proof is to solve approximate problems in arbitrary finite dimensional subspaces and to use the monotonicity property to carry out a limiting argument.

This theorem can be applied to yield an existence theorem for certain boundary value problems for nonlinear partial differential equations. It is supposed that the equation can be written in divergence form and that certain polynomial growth restrictions are met. The boundary value problem can then be given a variational formulation as an equation of the form $T(u) = f$ where T acts from a Sobolev space to its dual. This gives the so-called Hilbert space method when applied to linear differential operators. Monotonicity of T corresponds to an ellipticity requirement. However this is needed on the terms of all orders whereas, as in the linear case, ellipticity should only be required on the terms containing the highest order derivatives. This difficulty was overcome by combining a monotonicity requirement on the terms of highest order with compactness on the lower order terms. The compactness is obtained from the Sobolev embedding theorems. However, even in the absence of compact embeddings, a recent result of Browder [**4**] (not included in the book) shows that partial differential expressions give rise to mappings of monotone type, in this case the pseudomonotone operators [**1**].

If an elliptic boundary value problem has a linear part which possesses a Green's function, the problem can be transformed into an integral equation of Hammerstein type which can be written $u + KFu = 0$. Here K is a linear integral operator and F is a Nemitskii or substitution operator. In this context, operators of monotone type have been successfully employed, particularly, pseudomonotone operators and maximal (in the sense of graph inclusion) monotone operators.

In certain problems it is often natural to split the operator into two parts say as $T + S$, for example, linear part plus the nonlinear part. Clearly the range of $T + S$, $R(T + S)$ is contained in $R(T) + R(S)$. However, in a number of cases, for example T, S both maximal monotone plus some extra condition, these two sets are "almost equal" in that they have the same closures and the same interiors. These abstract results can be used to show that certain conditions necessary for boundary value problems to have solutions are close to being sufficient.

Certain free boundary value problems, problems in the presence of obstacles, and minimization problems with constraints, are best formulated as variational inequalities: given a closed, convex set K, find u in K such that

$$(u - v, T(u)) \geqslant 0 \quad \text{for all } v \text{ in } K.$$

Here again, using pseudomonotone operators seems to be the right generality in which to consider these problems. However, important results on existence and regularity of solutions were obtained for monotone operators. The article of Stampacchia [9] is an excellent introduction to these problems. The subject continues to expand.

Boundary value problems where some of the terms may grow rapidly have been called strongly nonlinear. One approach to these is to switch to an appropriate Orlicz-Sobolev space especially adapted to handle that growth. Another, due to Browder, which applies if these terms have the "correct" sign, allows one to remain in the framework of Sobolev spaces. This led to the study of new classes of operators of monotone type which are neither everywhere defined nor bounded. One of the tasks here is to weaken or remove certain technical assumptions. Recent results of Brézis and Browder [3] make a substantial step in this direction.

The present book covers all of the above topics and many more. It would be impossible to collect in a single volume all that is known about mappings of monotone type, so some selection is necessary. The authors have therefore only included applications to elliptic problems. Parabolic problems can be handled with minor modifications and some hints of this are given in the sections on further results and exercises at the end of each chapter. Similarly semigroups of nonlinear contractions, which are intimitely related to maximal monotone operators (at least, in a Hilbert space (e.g. [2])), are only mentioned in passing. However, whole books have been devoted to this topic.

A good knowledge of functional analysis is needed to read this book. A little is reviewed in the first chapter but, in my view, a few more definitions could have been included such as that of weakly convergent sequences. Also notation used on page 2 is not introduced until page 7.

The bulk of the book is about the various techniques that are being used with mappings of monotone type. However, some facts about compact operators are given, including a development of the topological degree (following Heinz). A useful section is included on Sobolev spaces but some technical proofs of embedding theorems might have been referenced rather than proved.

Each chapter ends with bibliographical comments. As some sections of the book closely follow the original papers these comments should have pointed out exactly who proved what, but often fail to do so.

There are the usual wealth of misprints and a few errors. For example, on page 3 a result of Browder is "proved". However, as soon as they deviate from Browder's correct proof they say "Since a normed linear space is separable if and only if its dual is, (Dunford and Schwartz p. 65) . . . ", which is false. The cited reference states the correct version. The proof on page 16 contains a slip (misprint?) and on page 281 it is stated that the truncation of a function in the Sobolev space $W^{m,p}$ also lies in the space: this holds only if $m = 1$ (or 0).

The book contains much material previously unavailable in book form. Some of the subjects are far from closed and developments have occurred since the book's publication. The book can well be read by someone who wishes to "get into" this subject. Whether it can be used in university courses, as the authors hope, is less clear.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0021/$02.25

Differentiation of real functions, by Andrew M. Bruckner, Lecture Notes in Math., vol. 659, Springer-Verlag, Berlin and New York, 1978, x + 246 pp., $12.00.

For most of us, the extent of our knowledge of the differentiation theory of real functions is quite limited. The standard information may be classified as follows:

(i) Derivatives share some of the properties of continuous functions, e.g., they have the intermediate value (Darboux) property.