there are some recent indications that these results can be obtained using less artificial and more direct methods (D. N. Shanbhag, 1979).

In spite of these reservations there is no doubt that the book has very successfully filled a longstanding gap in the literature and will be of immense usefulness to applied probabilists, statisticians and qualitatively oriented researchers dealing with probabilistic modelling who up until now were deprived of a comprehensive and carefully written compendium on elementary algebraic operations with one-dimensional random variables. It is hoped that a future edition of this book will incorporate a more detailed discussion of the algebra of multi-dimensional-random variables in view of the substantial demand and increase in applied probabilistic models based on multivariate distributions.

REFERENCES

B. Epstein, *Some applications of Mellin transforms in statistics*, Ann. Math. Stat. **19** (1948), 370–379.

H. Cramér, *Random variables and probability distributions*, Cambridge Tracts in Mathematics **36** (1937), Cambridge.

E. T. Whittaker and G. N. Watson, *A course in modern analysis*, Cambridge Univ. Press, London, 1915.

D. N. Shanbhag, *Personal communication*, 1979.

B. D. Carter and M. D. Springer, *The distribution of products, quotients and powers of independent H-function variates*, SIAM J. Appl. Math. **33** (1977), 542–558.

J. N. Lovett, *Evaluation of the H-function inversion integral*, Ph.D. thesis, University of Arkansas, 1977.

T. W. Hill, *On determining a distribution function known only by its moments and/or moment generating function*, Doctoral dissertation, Arizona State University, 1969.

A. Erdelyi, *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1954.

SAMUEL KOTZ

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0019/$02.00

Unitary group representations in physics, probability, and number theory, by George Mackey, Benjamin/Cummings, Reading, Mass., 1978, xiv + 402 pp., $19.50.

Even to one who does not wish to "buy" group representations as the end all or be all that it sometimes pretends to be, this is a very nice book. In particular, with Jauch's *Foundations of quantum mechanics* [1] on the one side and the present *Unitary group representations* by Mackey on the other, one has some forceful and interesting arm-chair reading in store. One should also keep Dirac's *Principles of quantum mechanics* [2] close at hand. In fact this reviewer was struck by and reminded of Dirac's elegant style and spare exposition when reading the present account by Mackey, even though the formats are different.

Mackey's book is the now published version of what are commonly called Mackey's Oxford notes [3] for lectures given there in 1966–1967. Although the treatment does include some discussion of applications to or perhaps more correctly relations to topics in probability, number theory, statistical mechanics, and the scattering theory of automorphic functons, its principal motivation remains that of quantum mechanics. As such, it effectively supersedes (although the author argues this point) his earlier book *Mathematical foundations of quantum mechanics* [4], which was based on lectures given at Harvard in 1960, and definitely contains (the author makes this clear) all material in the book *Induced representations of groups and quantum mechanics* [5], which was based on lectures given in the spring of 1967 in Pisa. On the other hand it does not supersede his *Chicago notes* [6], published in 1976 in revised form almost twice as extensive as the original lecture notes from the Chicago lectures given in 1955, and those with strong interests in group representations will want both the present book and the Chicago book.

Apparently we still do not have a really good introductory treatment of group representations. The recent treatise of Barut and Raczka *Theory of group representations and applications* [7] contains perhaps the most complete set of material dealing with the applications of group representations to physics.

This reviewer is not a group representor. My present and past interests in the subject stem from its relations to partial differential equations and to stochastic processes. One might also wish a review by a bona fide mathematical group representor, but most specialists in the subject would probably give an appreciative review anyway, and most of them will want the book in any case, unless perhaps they are satisfied with their tattered Xeroxed copy of the original Oxford notes. A review by a fully committed group representing genuine physicist would on the other hand be of considerable value to supplement the present review.

Group representations as a subject per se should be thought of as an abstraction of the Fourier methods for solving partial differential equations. In particular, its use is limited essentially to finding those solutions which may be had by separation of variables. While these statements are both oversimplifications, there is a lot of clarification obtained in admitting them. Some years ago when lecturing in Boulder Mackey at least acquiesced to this point of view as one that could not be immediately beaten down or stamped out or refuted in any general way.

Granting this point of view, and remembering how much harder separation of variables becomes when continuous spectra are encountered rather than just point spectra, when domains (e.g., not the whole space) appear in problems for which Fourier transforms are hard to calculate, when the groups are not just the integers or R^1, one can appreciate a method which maps t (the group) into $U_t = e^{itA}$ (the unitary representation) and then attempts to use the full force and generalization of Stone's Theorem and the Spectral Theorem to unscramble and analyze the group. On the other hand, if you are given the partial differential equation A (the operator) already, you are way ahead and should try to decompose it (separate variables) directly.

In the last five years the reviewer with others (including B. Misra and K. Goodrich) has found group representations handy in trying to understand certain stochastic processes of the white noise type. Mackey is also aware of this and it is one of the main motivations for his §§15 and 16, the only ones on probability theory included in the present book. So far, we have found that group representations provides a nice framework and suggests useful questions to ask and directions to take, but does not provide enough detail, and we are constantly forced back to "routine" Fourier analysis and approximation theory that seems to be as yet not done. Perhaps this is a limitation found for any general theory. Indeed, Stone's Theorem integrating the initial value problem $du/dt = iAt$, $u(0) = u_0$ as $u(t) = U_t u_0$ never really gives any precise information. For the latter one has to go further and calculate Fourier integrals or Green's functions.

A physicist colleague is also going to review this book. In his words, it is a bit dense. This may be reworded by saying that one already needs to know a fair amount about where group representations live and come from to be able to appreciate Mackey's present book. Again, it seems that someone should write a good (nonabstract) introductory treatment of the subject at, say, the senior-first-year-graduate level. Whether this can be done without getting lost in functional analysis is an open question.

How much more is there than in the original Oxford notes [3]? Not very much. Most is [3] verbatim. §30 on automorphic forms was corrected due to an error. Most sections are followed by a page or two of Notes and References. Although the latter are often restricted by the author to his own work or that of his students, when he goes further his selection of comments, references, and historical remarks are very interesting.

The portion on quantum mechanics is particularly good. As the author says, one can take it as an introduction to (the foundations of) quantum mechanics. In particular the reviewer liked the author's discussions of whether one should be in a Hilbert space and if so should the Hilbert space be complex or real, preoccupations the reviewer often shared in coffee discussions with J. M. Jauch. These are fundamental questions that have never been completely resolved. For further information and references see Jauch [**1**], [**8**] and Varadarajan [**9**].

Usually these two questions are taken up in that order. Perhaps they should be better separated. Both come from von Neumann's original conception [**10**] of modeling the experimental statements about a quantum system by means of the closed subspaces of a complex Hilbert space.

Mathematics has not been the same since, just as the advent of the physically formulated quantum mechanics some years earlier forever changed physics. Great progress has resulted, but at a price of lingering doubts and many profound questions. As is well known and about which much has been written, these questions arise in formulations that may appear as physical, mathematical, or philosophical problems. For a recent conceptual examination of some of them see d'Espagnat [**11**].

One arrives at a Hilbert space for a quantum logic because of a postulate of orthocomplementation which is desirable to accommodate yes-no experimental statements. One can assert therefore that this first question, that of why one should be in a Hilbert space, traces back to the so-called propositional calculus of yes-no experiments, which is in turn an attempt to describe a measuring process. Thus there is no absolute truth in the Hilbert space formulation, but until we understand better the theory and meanings of quantum mechanical measurement, the Hilbert space formulation is of great value as a working model.

The second question, that of whether the field should be real, complex, or quaternion, arises after one first (see Varadarajan [9]) makes some preliminary assumptions that reduce the possibilities to a division ring containing the reals. Thereafter one gives arguments showing that if one uses the real or the quaternion field for the Hilbert space formulation of quantum mechanics one in fact may equivalently use a complex Hilbert space. This is usually based on an additional physical assumption of a superselection rule. A superselection rule picks out a subset of unit rays that are declared to be not physically realizable. This gets rid of the problem for example in the real Hilbert space formulation of having "too many observables."

Mackey treats the latter by arguing that one has in fact only two more observables J_1 and J_2 which are square roots of the identity and which anticommute with i and with themselves. Other treatments yield just one J which is a square root of minus the identity. Any such superselection rule is a violation of the principle of superposition and should be carefully guided by experiment and physical theory and not just by mathematical duplication of i.

As a final comment on these questions, the reviewer would like to bring before the reader two other approaches that seem to be not well known: one old, one new.

In the marvelous little book by Temple [12] one finds a brief but rather good discussion (admittedly, based on dynamics). There the burden of which number field to use is placed on the correspondence principle. This goes further than the uncertainty principle and may be interpreted as allowing enough motion, which may in turn be seen as encumbering the physical description with a certain time reversal. Mathematically the latter shows up in the required existence and consideration of antiunitary operators in the model.

In the intriguing recent book by Brown [13], the case is made that in defining i by $x^2 = -1$ one is in fact defining the entity x in terms of itself according to $x = -1/x$. One is lead then to consider a four-valued logic in which a statement may be true, false, meaningless, or imaginary. This permits mathematical statements whose truth or untruth are perfectly decidable but which cannot be decided by the methods of reasoning to which one has previously restricted oneself.

Thus perhaps time reversal is the essential ingredient in understanding the complex scalars (the verbs, in Temple's words), and in a not unrelated way, an imaginary Boolean logic (like Brown's) can help explain the quantum logic and maybe even aid in the understanding of measurement-experiment theory.

Let us return to the book under review.

In his treatment of the foundations of quantum mechanics the author is on firm and well thought out ground and inserts confident statements such as "If only the mathematics (of quantum mechanics) were more tractable we could dispense with chemists altogether." This refers to the impossibility of exactly solving the Helium atom partial differential equations. It would be interesting to have a "solvable or unsolvable" statement explicitly worked out here by someone in Hamiltonian dynamics, i.e., "tractable or nontractable" clarified, as has been done for the classical three-body problem. Maybe it has been done somewhere.

The author is on less firm ground in the other sections on applications but it is still good reading. However, a statement as on the top of p. 153 "Our remarks apply with only obvious simple changes" (from the integers to the real line) shows some naiveté and includes three words (only, obvious, simple) which every mathematician knows may be cause for grave concern. On the other hand Mackey is clearly aware (in this situation) of the difficulties involved when natural orderings are not apparent. Perhaps it should be noted that the references given here to the probability literature are very incomplete.

To summarize, this is an extremely good book, written by a mathematician who is also a scientist and who is willing to make subjective statements to keep the theory alive and growing. It fills the bill in our current battles to revive the philosophy of mathematics as a part of a general scientific consciousness. It even passes the additional test of stating clearly certain open questions which remain in the theory and in the larger scientific investigations on which the theory may bear.

REFERENCES

1. J. M. Jauch, *Foundations of quantum mechanics*, Addison-Wesley, Reading, Mass., 1968.

2. P. A. M. Dirac, *Principles of quantum mechanics*, 4th ed., Clarendon Press, Oxford, 1958.

3. G. Mackey, *Group representations*, Oxford Notes, 1966–1967.

4. ______, *The mathematical foundations of quantum mechanics*, Benjamin, New York, 1963.

5. ______, *Induced representations of groups and quantum mechanics*, Benjamin, New York, Boringhieri, Torino, 1968.

6. ______, *The theory of unitary group representations*, Univ. of Chicago Press, Chicago, Ill., 1976.

7. A. Barut and R. Raczka, *Theory of group representations and applications*, Polish Sci. Publishers, Warsaw, 1977.

8. J. M. Jauch, *The mathematical structure of elementary quantum mechanics*, Sympos. on the Development of the Physicist's Conception of Nature in the Twentieth Century, (ed. J. Mehra), Reidel, Dordrecht, 1973.

9. V. S. Varadarajan, *The geometry of quantum theory*, VanNostrand, Princeton, N. J., 1968, 1970, originally Lecture Notes, Indian Statistical Institute, Calcutta, India, 1965.

10. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, Berlin, 1932.

11. B. d'Espagnat, *Conceptual foundations of quantum mechanics*, Benjamin, Menlo Park, Calif., 1971.

12. G. Temple, *The general principles of quantum theory*, Methuen Monographs, London, 1934.

13. G. Spencer Brown, *Laws of form*, Julian Press, New York, 1972.

KARL GUSTAFSON

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980

0002-9904/80/0000-0020/$01.75

Nonlinear mappings of monotone type, by Dan Pascali and Silviu Sburlan, Sythoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1978, x + 342 pp., $43.00.

In the study of nonlinear problems much use is made of compactness arguments. Particularly since the work of Leray and Schauder [5], the compact operators have been widely used in this study and new applications