BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980
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0002-9904/80/0000-0010/$02.00

Recursion-theoretic hierarchies, by Peter G. Hinman, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York, 1978, xii + 480 pp.

The turn of the century saw (amongst other things) the beginnings of Descriptive Set Theory. Faced with the increasing use of the all-powerful methods of Cantorian set theory and, in particular, the Axiom of Choice, mathematicians such as Baire, Borel, and Lebesgue began to develop more constructive approaches to analysis. Out of such investigations came the definition of the Borel sets, Souslin's theorem that a set of real numbers is Borel iff both it and its complement are analytic, the result that analytic sets are measurable and have the Baire property, and so on. Inherent in the notions of Borel set and analytic set (etc.) is the notion of hierarchy. Roughly speaking, a hierarchy is a classification of certain mathematical objects into levels, indexed by natural numbers, or possibly transfinite ordinal numbers as well. Objects appearing in levels low in the hierarchy are somehow more simple than those in higher levels: passage up through the hierarchy represents a gradually increasing complexity of the objects covered. For example, consider the Borel sets of real numbers. The standard definition of this class is that it is the smallest class of sets which contains all intervals and is closed under the formation of complements and countable unions. We can impose a hierarchy on this class by putting into the α-th level all sets which require a sequence of α applications of complementation and countable union to families of intervals for their construction. This hierarchy has precisely ω_1 levels, the level of any particular Borel set providing a measure of its complexity as a Borel set. Using this hierarchy we can, for instance, prove results about Borel sets by induction on the levels of the hierarchy.

Such hierarchies form a large part of the subject matter of this book. That this is so in a book published in the series "Perspectives in Mathematical Logic" (i.e. that the study of such hierarchies has fallen into the domain of the mathematical logician) arises from the marriage between classical Descriptive Set Theory and the developments in Logic which took place in the 1930s (and onwards). Logicians such as Church, Kleene, Turing, and Mostowski were looking at the notions of algorithmically calculable function, one function being recursive in another, the arithmetical and analytic hierarchies, and various sorts of definability in formal languages. By the mid 1950s, it became clear that classical Descriptive Set Theory and the above mentioned parts of Recursion Theory are really special cases of a single general theory of definability, this realisation being formally acknowledged by Addison's announcements in BAMS **61** (1955), 75; 171–172 (*Analogies in the Borel, Lusin, and Kleene hierarchies*. I, II). This general definability theory forms the starting point of the book.

The author is clearly addressing himself to more advanced students, a reasonable knowledge of analysis, topology, measure theory, set theory and logic being a prerequisite for the book. Moreover, the very nature of the subject gives the book the "heavy" look of a reference text as opposed to the gentler appearance of an "introductory text". Nevertheless, it is all there, and the suitably qualified reader should not have any problems with the development. Your humble (I have to say that for form's sake) reviewer hereby confesses his previous ignorance of large parts of the subject matter of the book, and was able to rectify matters by commencing at page 7 (where Chapter I commences) and proceeding through to page 457 (where the body of the text grinds to a final, tentative halt). (In other words, I think I was a typical reader of this book!)

Like Gaul, Hinman's book is divided into three parts. In Part A we meet the basic concepts of logic, ordinary recursion theory, definability, and the arithmetical and analytical hierarchies. Part B looks at the Analytical and Projective hierarchies in some detail, splitting the discussion into two parts–the first level, which can be handled quite well, and the other levels, where the going quickly gets tough. And in Part C the author branches out into the various generalisations of Recursion Theory which have been developed during more recent times.

Let us take a quick look at Part A. After some preliminaries concerning basic logic and set theory, topology and measure, and inductive definitions, the notions of primitive recursion, recursive functional, and recursive relation are defined, and the development taken through to the Normal Form Theo. rems. Unless the reader has some prior acquaintance with the early parts of Recursion Theory, the going will be very tough. The development is formal and nonintuitive, and is clearly designed to set up the required machinery in the most convenient form with the minimum of fuss. I would suggest that the novice reader has a look at the early parts of Roger's book *Theory of Recursive Functions and Effective Compatibility* (McGraw-Hill, 1967) before opening Hinman. Armed with a good intuition about recursive functions etc., Hinman's account can then be read without much trouble. (But note that Hinman uses the term "semi-recursive relation" instead of the older term "recursively-enumerable relation".) The first real hint of the connection between recursion theoretic hierarchies and classical Descriptive Set Theory comes next with the observation that if $R \subseteq {}^{\omega}\omega$ (where ${}^{\omega}\omega$ is the set of all ω-sequences of natural numbers), then R is open iff R is recursive in some real parameter β. So far, the development has been essentially that of the prerequisites for the later material. But with the start of Chapter III we get onto one of the main themes. The arithmetical and analytic hierarchies are defined and a number of technical properties of the hierarchies are proved.

The arithmetical hierarchy is defined thus. A relation (on $({}^{\omega}\omega)^n \times \omega^m$) is called Σ^0_0 (or Π^0_0) if it is recursive. Then, inductively, a relation is Σ^0_{n+1} (resp. Π^0_{n+1}) if it is determined by prefixing a Π^0_n (resp. Σ^0_n) relation with an existential (resp. universal) natural number quantifier. This gives a strictly increasing hierarchy of sets of relations. All members of the arithmetical hierarchy are Borel.

To obtain the analytical hierarchy, call a relation (on $({}^{\omega}\omega)^n \times \omega^m$) Σ^1_0 (or Π^1_0) if it is arithmetical and Σ^1_{n+1} (resp. Π^1_{n+1}) if it is determined by prefixing a Π^1_n (resp. Σ^1_n) relation with an existential (resp. universal) real number quantifier. (Throughout this book, the members of ${}^{\omega}\omega$ are the "reals", as is common in this subject.)

In both of the above definitions, if we allow arbitrary real number parameters to figure in the relations of level zero, we denote the enlarged hierarchy by using boldface type $\mathbf{\Sigma}^i_n$, $\mathbf{\Pi}^i_n$. The hierarchy $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ is of particular importance and is known as the *projective hierarchy*. Indeed, the first few levels of this hierarchy are familiar to all descriptive set theorists–though under a different guise. One immediately striking result is that the Borel relations are just those which are both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. Other questions of interest discussed in the book concern the question as to the relationship between the complexity of a set of reals and the complexity of the reals in that set.

Part A ends with brief discussions of definability in formal languages for arithmetic and arithmetical forcing, the latter being used to obtain results on the arithmetical hierarchy.

Part B deals with the analytical and projective hierarchies, and consists of just two chapters. The first of these deals with the sets $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. These are singled out for special consideration not only because they are more easily handled than higher levels in the hierarchy, but also because they are closely connected with notions in Descriptive Set Theory–the example that $\mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$ = Borel having already been mentioned.

The Borel hierarchy is defined as follows. Let $\mathbf{\Sigma}^0_0$ (or $\mathbf{\Pi}^0_0$) be the class of all clopen sets (i.e. subsets of $({}^{\omega}\omega)^m \times \omega^n$). For any ordinal $\alpha > 0$, let $\mathbf{\Sigma}^0_\alpha$ consist of all countable unions of members of $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, and let $\mathbf{\Pi}^0_\alpha$ consist of all complements of members of $\mathbf{\Sigma}^0_\alpha$. Then $\mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Pi}^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$, so the hierarchy stops growing at stage ω_1. It can be shown to be strictly increasing below ω_1, and clearly $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is the class of all Borel sets. For $n < \omega$, the sets $\mathbf{\Sigma}^0_n$, $\mathbf{\Pi}^0_n$ coincide with their previous definitions as members of the relativised arithmetical hierarchy.

Besides discussing the Borel hierarchy, this chapter provides a mine of useful results about Σ^1_1 and Π^1_1 sets from the point of view of cardinality, measure and category. One aspect of this which may well interest the newcomer is how techniques of recursion theory provide elegant proofs of standard results in Descriptive Set Theory, thereby exemplifying the benefits of the blending of the two subjects.

When we come to leave the familiarity of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ and venture further up the hierarchy, the going soon gets tough. Very little can be said without the assumption of additional axioms of set theory. Two (mutually inconsistent) such axioms have a profound effect on the projective hierarchy. The first of these to be considered is the Axiom of Constructibility ($V = L$). The key result here which effects the projective hierarchy is that, if $V = L$, the reals (i.e. ${}^{\omega}\omega$) have a well-ordering which is both Σ^1_2 and Π^1_2. The other axiom is the axiom of Projective Determinancy. The adoption of this axiom lends an altogether different flavor to the subject, and several appealing consequences can be obtained though Hinman does not go very far in this direction (other books on this very subject being already well under preparation). Nevertheless, he should stimulate the reader to look further into this fascinating branch of mathematics.

And so to Part C, where Hinman introduces us to various generalisations of recursion theory. Broadly speaking, there are two possibilities. Firstly, we can try to generalise our recursion theory from $({}^{\omega}\omega)^m \times \omega^n$ to $({}^{{}^{\omega}\omega}\omega)^\kappa \times ({}^{\omega}\omega)^m \times \omega^n$, and so on. Once the "correct" definition of "recursive" has been made, many of the results considered in the book can be generalised, and Hinman provides several such generalisations. The second generalisation arises when we try to replace ω by a larger ordinal number. Not all ordinals α admit a reasonable "recursion theory". Those that do are called "admissible ordinals". Much is now known about such ordinals, and they play an important role in Set Theory as well as Recursion Theory–indeed some of the arguments employed in recursion theory on admissible sets have a distinctly set-theoretic flavor!

It should be said that, despite our brief reference to this section given above, Part C occupies fully one half of the book, and contains a vast amount of material. Indeed, as Hinman says in his Preface, this is the material of which his volume was originally intended to consist!

So how did I find the volume? Well, let me first of all admit to being a reluctant reader (of any serious text); as well as one with a marked tendency to miss all sorts of errors. Consequently, I read the book in a fairly "shallow" fashion, and gained a fairly good impression of an area in which I am not at all expert. Armed with a reasonable foreknowledge of basic recursion theory and set theory as I was, I found the going not too bad. But the book is plainly intended for the more dedicated reader, with most proofs given in some detail. My feeling (prejudice?) here is that the lone reader may well find the going heavy (I would have, had I tried to read through it in depth), so that it would be preferable to couple the reading with a series of seminars or discussions on the material. There is a large selection of exercises, distributed throughout the text, some easy, some not so easy, and some with hints. So as a "standard text" the book stands very well indeed.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1980
© 1980 American Mathematical Society
0002-9904/80/0000-0011/$01.75

On uniformization of complex manifolds: The role of connections, by R. C. Gunning, Mathematical Notes no. 22, Princeton Univ. Press, Princeton, N. J., 1978, ii + 141 pp., $6.00.

In a celebrated inaugural address at Erlangen in 1872, Felix Klein defined geometry as the study of those properties of figures that remain invariant under a particular group of transformations. Thus, Euclidean geometry is the study of such properties as length, area, volume and angle which are all invariants of the group of Euclidean motions. In Klein's view, by considering a larger group one obtains a more general geometry. Thus Euclidean geometry is a special case of affine geometry. The latter in its turn is a special case of projective geometry. In any of these geometries, the group is relatively large. What Klein had in mind must be geometry of homogeneous spaces. For this reason, a homogeneous space G/H of a Lie group G is sometimes called a Klein space.