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# BOOK REVIEWS

*Symplectic groups*, by O. T. O'Meara, Mathematical Surveys, no. 16, Amer. Math. Soc., Providence, R. I., 1978, xi + 122 pp., $22.80.

The isomorphism theory of the classical groups over fields was initiated fifty years ago when Schreier and van der Waerden [7] determined the automorphisms of the projective special linear groups.

In 1951 Dieudonné [3] published an extensive study of the automorphisms of the classical groups and some of their large subgroups over possibly noncommutative fields. His method is to consider the involutions in a classical group $G$ over a sfield $F$ and the manner in which an automorphism $\Lambda$ of $G$ transforms these involutions among themselves. Some of these involutions are then related, either directly or in combinations, to one-dimensional subspaces in the underlying vector space $V$ on which the group $G$ acts, that is, to points in the projective space $\mathbf{P}(V)$ of $V$. Thus the automorphism $\Lambda$ is used to set up a bijective correspondence between the points in $\mathbf{P}(V)$ where, in the simplest case, the point associated with the involution $\sigma$ corresponds to the point associated with $\Lambda\sigma$. Then, after establishing that this correspondence is a projectivity, the Fundamental Theorem of Projective Geometry can be used to show that the automorphism is of some standard type such as $\Lambda: \varphi \mapsto \chi(\varphi)g\varphi g^{-1}$ for $\varphi \in G$, where $\chi$ is a homomorphism from $G$ into its center and $g$ is a semilinear automorphism of $V$. Variations of this technique have been used by other authors but it is limited to those classical groups and subgroups with an abundant supply of involutions. The situation circa 1963 has been summarized by Dieudonné [4].

To handle groups with few involutions, O'Meara in 1967 introduced a new approach where the points of the projective space $\mathbf{P}(V)$ are now associated with a different type of object in the group $G$. For example, if $G$ is linear or symplectic, a point can be associated with a nontrivial group of projective transvections in the following manner. A transvection is a linear transformation of $V$ of the form $\tau = \tau(a, \rho)$: $x \mapsto x + \rho(x)a$ where $a \in V$ and $\rho \in \mathrm{Hom}(V, F)$ with $\rho(a) = 0$; when $\tau \neq 1$, the residual space $(\tau - 1)V$ is the line $Fa$ in $V$. It is easier to work in the projective group $PG$ and here the point $Fa$ in $\mathbf{P}(V)$ is associated with the group of all projective transvections in $PG$ with residual line $Fa$ (plus the identity). The central problem now is to give a characterization of these groups of projective transvections and then deduce that their image under an automorphism $\Lambda$ is of the same form, so that one can set up a projectivity in $\mathbf{P}(V)$ and determine the nature of $\Lambda$. In orthogonal groups there may be no transvections so these are replaced by plane rotations; one-dimensional subspaces of $V$ are then obtained by intersecting planes. This new approach has been extremely successful, and in a sequence of papers over the last twelve years, O'Meara and his colleagues have developed an extensive theory of isomorphisms between subgroups of classical groups over fields and, more generally, studied arithmetic questions for congruence subgroups defined over integral domains. Naturally, there has been considerable improvement in sophistication over the earlier papers and in [6] O'Meara gives a self-contained account of these methods, along with related material on generators and structure, for the linear group.

Since this monograph was written, however, another substantial improvement has been made and further groups which had originally escaped, particularly groups over noncommutative fields, can now be handled. The improvement lies in the way in which it is shown that the projective image of $\Lambda\tau(a, \rho)$ is a projective transvection. The earlier CDC-method was group theoretic, involving a study of centralizers and derived groups. The new method is more geometric, using an intricate consideration of the action of $\Lambda\tau(a, \rho)$ on points in $\mathbf{P}(V)$. The book under review gives an essentially self-contained treatment of these new methods by solving not only the basic isomorphism problems for symplectic groups but also for the larger group of symplectic collineations. As with the companion work on linear groups, there are sections on the generators and structure of the symplectic groups. These two books now provide the best introduction to the isomorphism theory of classical groups over fields and integral domains and certainly will be a starting point for further research extending these results to rings with zero divisors or incorporating them into the general theory of algebraic groups. In these directions, recent work over local rings using the older involution approach can be found in McDonald [5], and in [1] Borel and Tits give a treatment for some algebraic groups, studying not only isomorphisms but also homomorphisms. There is doubtless much more that can be done in this general domain relating groups and geometry.

To give some idea of the generality of the results now available, we will quote two of the theorems. For $i = 1,2$, let $\mathfrak{o}_i$ be integral domains with quotient fields $F_i$ and denote by $\mathrm{Sp}(n_i, \mathfrak{o}_i)$ the subgroups of integral transformations in the symplectic groups $\mathrm{Sp}(V_i)$ acting on vector spaces $V_i$ over $F_i$ of even dimension $n_i \geqslant 4$ endowed with nondegenerate alternating forms.

THEOREM. $\mathrm{Sp}(n_1, \mathfrak{o}_1) \cong \mathrm{Sp}(n_2, \mathfrak{o}_2) \Leftrightarrow n_1 = n_2$ *and* $\mathfrak{o}_1 \cong \mathfrak{o}_2$.

Furthermore, for $i = 1,2$, let $M_i$ be $\mathfrak{o}_i$-modules in $V_i$ with $F_iM_i = V_i$ and which are contained in some free $\mathfrak{o}_i$-modules on $V_i$. Define $\mathrm{Sp}(M_i) = \mathrm{GL}(M_i) \cap \mathrm{Sp}(V_i)$ and let $\mathrm{Sp}(M_i; \mathfrak{a}_i)$ be the congruence subgroups with respect to the nonzero ideals $\mathfrak{a}_i$ in $\mathfrak{o}_i$. Denote by $\Gamma\mathrm{Sp}(V_i)$ the groups of symplectic collinear transformations on $V_i$, namely, the groups of semilinear bijections of $V_i$ which preserve the underlying alternating forms up to scalar multiples and field automorphisms.

THEOREM. *Let $G_i$, $i = 1,2$, be groups with*

$$\mathrm{PSp}(M_i; \mathfrak{a}_i) \subseteq G_i \subseteq \mathrm{P\Gamma Sp}(V_i).$$

*Then, except when the fields $F_i$ are perfect with characteristic two and $n_i = 4$, an isomorphism $\Lambda$: $G_1 \to G_2$ has the form*

$$\Lambda\varphi = g\varphi g^{-1} \quad \forall\varphi \in G_1$$

*for a unique projective symplectic collinear transformation $g$ of $V_1$ onto $V_2$.*

The exceptional situation for perfect fields of characteristic two when

$n_1 = n_2 = 4$ is also analyzed. There is now an extra isomorphism corresponding to the graph automorphism of the Dynkin diagram for $C_2$.

Much of the material in this book, including the two theorems above, is more general than previously published results in the area. Recently, Callan [2] has applied the same method to unitary groups over noncommutative domains possessing a division ring of quotients, assuming the underlying Witt index is at least three. The objectives of the book and the basic methods are very clearly presented. No problems are included, but then none are needed, for the best way to fully understand some of the more intricate proofs is to break them into pieces, as does the author, and work out the separate details for oneself.

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*Fundamentals of decision analysis*, by Irving H. LaValle, Holt, Rinehart and Winston, New York, Chicago, San Francisco, Atlanta, Dallas, Montreal, Toronto, London, Sydney, 1978, xiii + 626 pp.

To know the rules is not the same as to know how to play the game. As in chess or tennis, so in decision analysis. Decision analysis is applied decision theory, or how to make decisions that are consistent with the choices, information, and preferences of the decision maker. Decision analysis is both a language and philosophy for decision making and a practical procedure for arriving at decisions. The procedure consists of analyzing (Latin: loosening back) the decision problem into its choice, information, and preference component parts, which can then be judgmentally assessed by the decision maker and combined by logic to allow a consistent course of action. To see why this book would be more appropriately titled *Fundamentals of statistical decision theory*, we must consider the present state of decision analysis in more detail.

The domain of decision analysis is shown graphically in Figure 1. The first three rows represent the three elements of formulation that we have dis-