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THE QUEER DIFFERENTIAL EQUATIONS FOR ADIABATIC COMPRESSION OF PLASMA

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This announcement presents some results on the "Queer Differential Equations" (QDE) of adiabatically evolving plasma equilibria. These are nonlinear differential-functional equations of the form $\Delta\psi = F(V, \psi, \psi', \psi'')$, where $V = V(\psi)$ is the volume (area) inside the levelsets $\psi(\mathbf{r}) = \psi$, a constant, and the derivatives on the right hand side are with respect to the dependent variable V, e.g., $\psi'(V)$.

We describe properties of microcanonical averages and their derivatives and a simple example of the nonlinear problem. An existence and uniqueness theorem is given for the associated linearized problem, which is also a functional-differential equation. Finally we will mention an isoperimetric problem related to the geometry of QDE's.

A few of these results are in [1] and will be further expounded in [2]–[4]. For the initial development of this problem and the relevant physics see [5]–[7] and the references therein. For a related problem, see also [8], [9].

Averages. S is a simple domain in the plane and $z = \psi(\mathbf{r})$, $\mathbf{r} \in \bar{S}$, a surface such that $\psi \in C^{n,\alpha}(\bar{S})$, $n \geq 2$, $0 \leq \alpha < 1$. Assume that the level lines $\psi(\mathbf{r}) = \psi$ are simple and $\nabla\psi = \mathbf{0}$ at only one point $\mathbf{r}_0 \in S$, $\psi(\mathbf{r}_0) = \psi_0$. Let ∂S be the level line $\psi(\mathbf{r}) = \psi_1$ $(>\psi_0)$. $V = V(\psi)$ is defined to be the area inside the curve $\psi(\mathbf{r}) = \psi$ and $\psi(V)$ is the inverse of $V(\psi)$.

$$V'(\psi) = \oint_\psi |\nabla\psi|^{-1}\, ds$$

is continuous in the interval $I = (\psi_0, \psi_1]$.

The function $V(\mathbf{r}) = V(\psi(\mathbf{r}))$ is useful; it is C^1 in $\bar{S}\backslash\mathbf{r}_0$.

THEOREM 1. *Let ψ be as described above and assume that $\mathbf{r}_0$ is an elliptic critical point. Then $V(\mathbf{r}) = V(\psi(\mathbf{r}))$ is also in $C^{n,\alpha}$ and $\mathbf{r}_0$ is an elliptic critical point for $V = V(\mathbf{r})$.*

Consider $g \in C^{m,\alpha}(\bar{S})$ and define on I,

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$$\tilde{g}(\psi) = \oint_\psi g(\mathbf{r}) |\nabla\psi|^{-1}\, ds.$$

Using Green's formula the differentiation with respect to level lines is given by

$$\tilde{g}'(\psi) = \frac{d}{d\psi} \oint_{\psi(\mathbf{r})=\psi} g(\mathbf{r}) \frac{ds}{|\nabla\psi|} = \oint_\psi \nabla\psi \left\{ \frac{g}{|\nabla\psi|^2} \nabla\psi \right\} \frac{ds}{|\nabla\psi|}. \tag{1}$$

When $g = 1$ this allows us to compute $V^{(n)}(\psi)$. The integrand in (1) is oscillatory with an unbounded amplitude when $\psi \longrightarrow \psi_0$ (i.e. $\mathbf{r} \longrightarrow \mathbf{r}_0$) but the behaviour there can be adequately described by using a coordinate-transformation and Taylor series in the neighborhood of the critical point. In fact we have:

THEOREM 2. *Let $\psi \in C^{n,\alpha}$ be as in Theorem 1 and $g \in C^{m,\alpha}$, then $\tilde{g} \in C^{p,\alpha/2}(I)$, where $p = \min\{m, n-1\}$. If $k \leqslant p$ and $r = \max\{k + 1 - p/2 - \alpha/2, \alpha/2\}$, then $(\psi - \psi_0)^r \tilde{g}^{(k)}(\psi) \longrightarrow 0$ as $\psi \longrightarrow \psi_0$. If $m = 2k$ and $n = 2k + 2$, then $\tilde{g}^{(k)}(\psi)$ can be defined continuously at $\psi = \psi_0$.*

Finally notice that $\oint_V |\nabla V|^{-1}\, ds = 1$ and that the microcanonical average of g is given by

$$\langle g \rangle_V = \bar{g}(V) = \oint_\psi g(\mathbf{r}) \frac{ds}{|\nabla\psi|} \Big/ \oint_\psi \frac{ds}{|\nabla\psi|} = \oint_V g(\mathbf{r}) \frac{ds}{|\nabla V|} \tag{2}$$

Nonlinear problems. Consider on S

$$\Delta\psi = -\psi'', \tag{3}$$

and admit for the moment only solutions with simple level lines. As boundary data we specify $\psi(\mathbf{r}) = \psi_1$, a given constant on ∂S. This is not sufficient for a well-posed problem. Using $\Delta\psi = \psi'\Delta V + \psi''|\nabla V|^2$ and (3) there follows

$$\Delta V/(1 + |\nabla V|^2) = h(V), \qquad V|_{\partial S} = V_1, \tag{4}$$

$$\psi''/\psi' = -h(V), \quad \psi(V_1) = \psi_1, \tag{5}$$

where $V_1 = V(\psi_1)$ is the area of S. We can now ask if there exists an h in some properly restricted class of functions, such that the solution of (4), $V = V(\mathbf{r})$, has simple level lines and satisfies the constraint $\oint_V ds/|\nabla V| = 1$. The solution of (3) is then given by $\psi(\mathbf{r}) = \psi(V(\mathbf{r}))$ where $\psi = \psi(V)$ solves (5). It is now clear that other data is needed for (5), and physically it is natural to give $\psi(0) = \psi_0$.

No existence theorems are known for classical solutions of (3) for nontrivial geometries, but for simpler QDE's a few results are known.

THEOREM 3. *Let $\psi \in C^{2,\alpha}(S)$, $0 < \alpha < 1$, have simple level lines and a nondegenerate critical point and solve $\Delta\psi = F(V, \psi)$, where $F \in C^\infty$. Then $\psi \in C^\infty$.*

THEOREM 4. *Assume that T is a plane annular domain. Let $\psi \in C^{2,\alpha}(T)$, $0 < \alpha < 1$, have simple level lines and $|\nabla\psi| > 0$ and solve $\Delta\psi = F(V, \psi, \psi')$ where $F \in C^\infty$. Then $\psi \in C^\infty$.*

These follow from previous theorems and well-known theorems on the regularity of solutions of elliptic partial differential equations.

THEOREM 5. *Let S be convex, let $\Delta\phi = 1$ and $\phi|_{\partial S} = 0$, and let ϕ have simple convex level lines with a nondegenerate critical point. Assume that $F = F(V)$ is C^1 and $F(0) \neq 0$. Then if F' is sufficiently small,*

$$\Delta\psi = F(V), \qquad \psi|_{\partial S} = \psi_1 \tag{6}$$

has a solution in C^2 with the same geometric properties as ϕ.

This is proven by the iteration $\Delta\psi_m = F(V_{m-1})$, $V_0(\mathbf{r}) = V(\phi(\mathbf{r}))$.

Linearized problem. The perturbed problem for $\Delta\psi = F(V, \psi, \psi', \psi'')$ or (3) is also a QDE

$$L_2\phi = -L_1\bar{\phi} \tag{7}$$

where $\phi = \partial\psi/\partial t$ and $\bar{\phi}(V) = \langle\phi\rangle_V$. L_2 is a second order elliptic operator, $L_2 = \Delta - P$, and L_1 is a second order ordinary differential operator. P and the coefficients of L_1 are given in terms of ψ, which is assumed to have a few derivatives.

What distinguishes the QDE (7) from (3) is that the contours used to determine $\bar{\phi}$ from ϕ are now assumed to be given; they are the level lines of $\psi = \psi(\mathbf{r})$.

Then without being too specific,

THEOREM 6. *Assume that L_2 and L_1 are negative definite and have sufficiently smooth coefficients. Let the level lines used to determine $\bar{\phi}$ be simple, have an elliptic critical point and several derivatives. Specify the boundary values $\phi|_{\partial S} = \phi_1$ and $\bar{\phi}(0) = \phi_0$. Then there exists a unique classical solution for* (7).

The proof uses the Green function of L_2, $G = G(\mathbf{r}, \mathbf{r}')$, and its double average $\hat{G}(V, V') = \langle\langle G\rangle_V\rangle_{V'}$, and

THEOREM 7. *The double average of a parametrix (or Green function) for a second order elliptic partial differential operator is a parametrix for a second order ordinary differential operator L_V, i.e. $L_V\hat{G} = I + A$. I is the identity and A is a compact integral operator.*

Using this, some manipulation transforms (7) into a Fredholm integral equation of the second kind for $\bar{\phi}$.

Isoperimetric problem. The function $K(V) = \langle |\nabla V|^2 \rangle_V$ can be interpreted as the capacity of the (given) distribution of level lines on S. It is a purely geometrical quantity.

Let $\phi \in C^1$ and consider the functional

$$\widetilde{\Phi}[\psi] = \Phi[V] = \int_0^{V_1} \phi(K(V))\, dV. \tag{8}$$

The isoperimetric problem we are interested in is to find the extrema of Φ subject to the constraint $\oint |\nabla V|^{-1}\, ds = 1$. In particular, when $\phi(t) = (1 + t)^{-1}$, then the "Euler's equation" for (8) is (4) with an unknown h which has to be chosen so that the constraint is satisfied.

In all the arguments above we have assumed a simple geometry, but non-simple level lines are actually more relevant for physical applications and extensive numerical schemes have been developed by H. Grad and coworkers to study these cases; see the cited references.

The functional in (8) can be reinterpreted so as to include the more complicated geometries. In fact for the special case of ϕ mentioned above, finding the supremum of $\Phi[V]$ over a properly constrained class of V's would then correspond to finding the solution of (4) with the simplest geometry.

This variational formulation will be used to study examples of bifurcation, exchange of stability and evolution into more complicated geometries.

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