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DISPLACEMENT RANKS OF A MATRIX[1]

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The solution of many problems in physics and engineering reduces ultimately to the solution of linear equations of the form $Ra = m$, where R and m are given $N \times N$ and $N \times 1$ matrices and a is to be determined. Here our concern is with the fact that it generally takes $O(N^3)$ computations (one computation being the multiplication of two real numbers) to do this, and this might be a substantial burden if N is large or if the problem has to be repeated with different R and m. For such reasons, one often seeks to impose more structure on the matrices R. In many problems we have an underlying stationarity or homogeneity (invariance under displacements in time or space) property that often leads to the matrix R being *Toeplitz* (i.e., with elements of the form R_{i-j}). Now it is known that Toeplitz matrices can be inverted with $O(N^2)$ (or even $O(N \log^2 N)$) multiplications, which can be considerable simplification. However, even if the physical problem has an underlying stationarity property, it still happens that in the course of the analysis the coefficient matrix R turns out to be non-Toeplitz, though in some sense close to Toeplitz. For example R may be the inverse of a Toeplitz matrix, or the product of two rectangular Toeplitz matrices (as arises often in least-squares theory), or an asymptotically Toeplitz matrix ($R_{ij} \longrightarrow R_{i-j}$ as $i, j \longrightarrow \infty$). It seems unreasonable that equations with such non-Toeplitz matrices should require $O(N^3)$ operations for their solution, but this question does not seem to have been systematically explored before.

Motivated by a number of related results on the solution of certain nonlinear (Riccati- and Chandrasekhar-type) differential equations arising in some least-squares estimation problems ([1]-[3]), we have been able to provide some answers to the above question and also obtain some extensions. Roughly speaking, with an $N \times N$ matrix R we are able to associate an integer α, $1 \leqslant \alpha \leqslant N$, that seems to provide a nice measure of how close R is to being Toeplitz; moreover, we have shown that a matrix with index α can be inverted with (about) α times as much computations as required for a Toeplitz matrix.

To make these statements more precise, we introduce two so-called *displacement ranks* of a matrix.

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DEFINITION. The ($\pm$)-*displacement ranks* of an $N \times N$ matrix R are the smallest integers $\alpha_\pm(R)$ such that we can write

$$R = \sum_1^{\alpha+} L_i U_i \quad \left(\text{or } R = \sum_1^{\alpha-} \mathcal{U}_i \mathcal{L}_i \right) \tag{1}$$

for some lower-triangular Toeplitz matrices $\{L_i\}$ (or $\{\mathcal{L}_i\}$) and some upper-triangular Toeplitz matrices $\{U_i\}$ (or $\{\mathcal{U}_i\}$).

THEOREM 1. *The* ($\pm$)-*displacement rank of a matrix is equal to the* ($\mp$)-*displacement rank of its inverse, i.e.,* $\alpha_+(R) = \alpha_-(R^{-1})$ *and* $\alpha_-(R) = \alpha_+(R^{-1})$.

EXAMPLE. If T is a symmetric Toeplitz matrix, then the representations

$$T = T_+ \cdot I + I \cdot T_+' = I \cdot T_+ + T_+' \cdot I$$

where T_+ is the lower-triangular part of T, show that $\alpha_\pm(T)$ are not greater than 2, and we shall show presently that (unless T is diagonal or zero) $\alpha_+(T) = 2 = \alpha_-(T)$. Therefore according to the theorem we must have $\alpha_-(T^{-1}) = 2 = \alpha_+(T^{-1})$, and in fact it is known (see, e.g., [5], [6]) that there exist lower-triangular Toeplitz matrices $\mathcal{A}$ and $\mathcal{B}$ such that $T^{-1} = \mathcal{B}'\mathcal{B} - \mathcal{A}'\mathcal{A}$.

LEMMA 1. *Alternative characterization of displacement ranks. The* ($\pm$)-*displacement ranks can be computed as*

$$\alpha_+(R) = \text{rank}\{R - ZRZ'\}, \ \alpha_-(R) = \text{rank}\{R - Z'RZ\},$$

where the prime denotes transpose and Z *is the "lower-shift" matrix consisting of* 1's *along the first subdiagonal and zeros elsewhere.*

Writing out $R - ZRZ'$ and $R - Z'RZ$ for 3×3 matrices will explain the reason for the name displacement rank and will also show that $|\alpha_+ - \alpha_-| \leq 2$. It is also worthwhile to check that $\alpha_\pm(T) = 2$ by applying Lemma 1. Note also that the rank is 2 even under numerical perturbations in the elements, provided the Toeplitz structure is respected. Similar statements hold for representations as in (1).

LEMMA 2. *A functional equation. Given column vectors* $\{x_i, y_i\}$, *the unique solution of*

$$R - ZRZ' = \sum_1^{\alpha} x_i y_i' \tag{2a}$$

is

$$R = \sum_1^{\alpha} L(x_i) U(y_i'), \tag{2b}$$

where $L(x)$ *denotes a lower-triangular Toeplitz matrix whose first column is* x *(this completely specifies the matrix), and* $U(y')$ *denotes an upper-triangular Toeplitz matrix whose first row is* y'. [*There is a similar result for* $R - Z'RZ$.]

Lemma 2, which is easy to check, can be used in a fairly obvious way to prove Lemma 1, which will now be used to prove Theorem 1.

PROOF OF THEOREM 1. Let $\rho\{A\}$ denote the rank of A. Then we note that since rank is unaffected by multiplication by a nonsingular matrix,

$$\alpha_-(R^{-1}) = \rho\{R^{-1} - Z'R^{-1}Z\} = \rho\{(R^{-1} - Z'R^{-1}Z)R\} = \rho\{I - Z'R^{-1}ZR\}.$$

Now by a well-known matrix result that $\rho\{I - AB\} = \rho\{I - BA\}$, we can continue the above chain as

$$\begin{aligned}\alpha_-(R^{-1}) &= \rho\{I - ZRZ'R^{-1}\} \\ &= \rho\{(I - ZRZ'R^{-1})R\} = \rho\{R - ZRZ'\} = \alpha_+(R).\end{aligned} \tag{3}$$

A similar argument will establish that $\alpha_+(R^{-1}) = \alpha_-(R)$. □

This simple proof shows that in fact the result of Theorem 1 is quite general. Thus it depends very little on the nature of the entries of R, as long as they are such that the cited rank properties still hold. For example, the entries of R could be matrices themselves. It is also rather striking that the proof does not depend upon what the matrix Z actually is. We defined it above as a lower-shift matrix because we wish to focus (see Theorem 2 below) on relations to Toeplitz matrices, which are (almost) invariant under a shift. But other emphases can be accommodated by choosing Z differently. For example, we could focus on relations to 'periodic' matrices by choosing Z as the 'unit circulant matrix'; 'Hankel' matrices could be handled by forming $R - ZRZ$ and $R - Z'RZ'$. The results can also be adapted to treat integral operators (see [4]). [Here we only mention that the displacement rank of an integral operator with kernel $K(t, s)$ can be defined as the smallest α such that we can write

$$(\partial/\partial t + \partial/\partial s)K(t, s) = \Sigma_1^\alpha\, \phi_i(t)\psi_i(s),$$

for some $\{\phi_i, \psi_i\}$ (compare with Lemma 1).]

Lemma 2 shows that representations of the form (1) can be obtained with many choices of vectors $\{x_i, y_i\}$ and many choices of α. The smallest possible value of α will be the rank of $R - ZRZ'$, but unless we have some a priori information on R, this rank may not be easy to determine by direct numerical evaluation. The result of Theorem 2 is helpful in this connection.

THEOREM 2. *Suppose that we have a representation of R as*

$$R = \sum_1^\alpha L(x_i)U(y_i'), \tag{4}$$

not necessarily a minimal one (i.e., $\alpha \geqslant \alpha_+(R)$). Suppose also that all the leading minors of R are nonzero. Then there exists an algorithm for computing

R^{-1} in the form

$$R^{-1} = \sum_{1}^{\alpha} U(a_i')L(b_i) \tag{5}$$

with of the order of $N^2\alpha$ multiplications.

The significance of this result is that in the actual applications, we might be satisfied with representations (4) that are "reasonable approximations" to R. The reduction in computational effort gives us some flexibility in trying to find a 'good' solution by varying R or varying α.

We may note also that the representation (5) for R^{-1} allows us to write bilinear forms $x'R^{-1}y$ as $\Sigma_1^\alpha\ (L(a_i)x)'(L(b_i)y)$, the significance being that (because $L(b_i)$ is Toeplitz) $L(b_i)y$ is just the convolution of b_i and y. Therefore *FFT* techniques can be used to find $L(b_i)y$ in $O(N \log N)$ operations [7] and consequently $x'R^{-1}y$ can be evaluated in $O(\alpha N \log N)$ operations as compared to $O(N^2)$ for an arbitrary matrix R^{-1}.

The proof of Theorem 2 is constructive, and gives a recursive procedure for successively inverting the principal submatrices of R. In fact, it is a striking fact that the algorithm has the same 'form' as the Levinson-Trench-Szegö algorithms (see, e.g., [6]) for inverting a Toeplitz matrix–only the dimensions of certain variables and the values of certain parameters are determined differently, in a way that depends on the actual form of the representation (1). These results and further extensions (e.g., to higher-order displacement ranks and to integral operators), and applications to the computation of least-squares predictors (conditional means) and likelihood ratios (Radon-Nikodym) derivatives for Gaussian processes will be described elsewhere.

In connection with Theorem 2, Dr. D. Yun of IBM and a referee have noted that by judicious use of *FFT* ideas, Toeplitz equations can be solved with $O(N \log^2 N)$ operations (cf. the *HGCD* algorithm in §8.9 of [7]). These results can also be suitably extended to matrices of the form (5), see e.g. [8].

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