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THE L^2-INDEX THEOREM FOR HOMOGENEOUS SPACES

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The geometric realization of the irreducible square integrable representations for semisimple Lie groups (cf. [3], [6]) and also for nilpotent Lie groups [5] suggests that, as a general phenomenon, such representations should appear as L^2-kernels of invariant elliptic operators. One basic problem in this respect is to decide when such a kernel is nonzero. In the compact case the basic tool for this, used in the Borel-Weil-Bott approach, is the Hirzebruch-Riemann-Roch theorem. In the noncompact case one needs an analogue of the index theorem of Atiyah-Singer [2] for noncompact manifolds. When G possesses a discrete cocompact subgroup, the L^2-index theorem for covering spaces of [1] and [7] provides the required analogue. Our purpose here is to give a general index theorem for homogeneous spaces of arbitrary connected unimodular Lie groups, essentially based on the index theorem for foliations [4].

So let G be a connected unimodular Lie group, and let H be a closed subgroup of G which contains the center Z of G and such that H/Z is compact. Let χ be a character of Z, and let E, F be finite-dimensional unitary representations of H whose restrictions to Z are given by χ. Denote by $\mathbf{E}$, $\mathbf{F}$ the corresponding (invariant) induced bundles on the homogeneous space $M = G/H$, and let D be an invariant elliptic differential operator from $\mathbf{E}$ to $\mathbf{F}$. The representation of G in the kernel of D in $L^2(M, \mathbf{E})$ is square integrable modulo the center of G (see [4]), though not necessarily irreducible. Its formal degree deg(Ker D) (as defined in [4]) is always finite, so that the analytical index of D can be defined as

$$\operatorname{Ind}(D) = \deg(\operatorname{Ker} D) - \deg(\operatorname{Ker} D^*).$$

We now describe the topological index of D. Let $\mathfrak{M}$ be an Ad H invariant supplement for Lie (H) in Lie (G). We can assume, dividing by Ker χ, that H is compact. The principal symbol of D defines an element σ_D of $K_H(\mathfrak{M}^*)$, the equivariant K-theory with compact support of the dual vector space $\mathfrak{M}^*$ of $\mathfrak{M}$. Using the Thom isomorphism at the level of the rational cohomology of the classifying space for H, one gets a natural map τ from $K_H(\mathfrak{M}^*)$ to the completion $(R(H) \otimes \mathbf{Q})^\wedge$ of the representation ring of H. Let then $H_G^*(M, \mathbf{R})$ be the cohomology ring of G-invariant differential forms on the homogeneous space M.

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For each finite-dimensional representation E of H, the curvature $2\pi i\Omega_E$ of the corresponding induced bundle on M (with respect to the invariant connection defined by $\mathcal{M}$) allows to define an element $\omega(E)$ of $H_G^*(M, \mathbf{R})$ represented by the invariant differential form:

$$\omega(E) = \mathrm{Trace}(\exp(\Omega_E)).$$

This map ω extends to a homomorphism of $(R(H) \otimes \mathbf{Q})^\wedge$ to $H_G^*(M, \mathbf{R})$. Put

$$\mathrm{ch}\, D = \omega(\tau(\sigma_D)).$$

Further, let $2\pi i\Omega$ be the curvature of the complexification of the tangent bundle of M (with respect to the invariant connection given by $\mathcal{M}$), and let

$$\mathrm{Td}(\mathcal{M}_{\mathbf{C}}) = \det \frac{\Omega}{1 - \exp(-\Omega)}$$

be the corresponding Todd class. Let dg be the chosen invariant volume element on G/Z, let dh be the invariant volume element of total mass 1 on H/Z and let $v \in \Lambda^n(\mathcal{M})$ be such that $\langle dg/dh, v\rangle = 1$. With the above notations we have:

THEOREM. $\deg(\mathrm{Ker}\, D) - \deg(\mathrm{Ker}\, D^*) = \langle \mathrm{ch}(D)\mathrm{Td}(\mathcal{M}_{\mathbf{C}}), v\rangle$.

When G has a discrete torsion free cocompact subgroup Γ the theorem follows from the L^2-index theorem for covering spaces of [1], [7] applied to the action of Γ on G/H. In general (for instance for the obvious semidirect product of the Heisenberg group by $SL(2, \mathbf{R})$ or for general nilpotent Lie groups) such a Γ does not exist in G. As a substitute for [1] we use the index theorem for foliations [4].

To construct the foliation we let V be the compact manifold $\Gamma\backslash SL(\mathfrak{G})$, where $\mathfrak{G}$ is the Lie algebra of G and Γ is a discrete torsion free cocompact subgroup of $SL(\mathfrak{G})$. The adjoint representation of G on $\mathfrak{G}$ determines a Lie isomorphism of G/Z into $SL(\mathfrak{G})$ and hence an action of G/Z on V. Dividing V by the action of H gives a compact manifold, foliated by the quotient Φ of the action of G/Z on V. The only serious problem which arises in applying to this foliation the result of [4] is that the bundles $\mathbf{E}$, $\mathbf{F}$ on G/H only define "projective bundles" on V/H so that a slight generalization of [4] is required.

At first glance the natural approach to this index theorem seemed to be: (1) the development of a G-invariant pseudo-differential calculus on the homogeneous space M; (2) the use of the heat equation method to compute the index of special operators; (3) K-theory arguments on $K_H(\mathcal{M}^*)$ to reach the general case. In each of these steps our method turns out to be more effective: in (1) and (2) because the local triviality of the foliation Φ allows us to use the well-established local computations, and in (3) because the K-theory of $T\Phi$ (the tangent bundle to the leaves) is indeed generated by the symbol of the signature operator, which is not the case for $K_H(\mathcal{M}^*)$, as an $R(H)$-module.

Details and applications will appear elsewhere.

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