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THE RESTRICTED SIMPLE LIE ALGEBRAS WITH A TWO-DIMENSIONAL CARTAN SUBALGEBRA[1]

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The classification of the finite-dimensional simple Lie algebras over an algebraically closed field of prime characteristic p is still an open problem after 40 years of investigation (going back to the pioneering work of Jacobson and Zassenhaus). The problem can be made more tractable by considering only restricted Lie algebras (i.e., Lie p-algebras), these being Lie algebras with an extra mapping $x \longmapsto x^p$ satisfying certain hypotheses, in particular $(\mathrm{ad}\, x)^p = \mathrm{ad}\, x^p$ (see [3, p. 187]); throughout the subject's history this restriction has proved fruitful, being a good indicator of the general shape of nonrestricted results as well. It is also customary to exclude some small p. But even with these two hypotheses the problem remains very much open, although progress to date (some of it mentioned below) and the new contribution announced here, namely, the classification of the algebras of the title, give cause for optimism.

From now on let L denote a finite-dimensional simple restricted Lie algebra over an algebraically closed field F of characteristic $p > 7$ (some of the results cited below hold in more generality). The known L are of two kinds: either classical, i.e., algebras of the usual types $A_n, \ldots, G_2$, these being analogues of the finite-dimensional simple Lie algebras over $\mathbf{C}$, or of Cartan type, i.e., one of the algebras W_n, S_n, H_n or K_n, these being analogues (previously known but first described in this way by Kostrikin and Šafarevič [7]) of the infinite-dimensional simple Lie algebras over $\mathbf{C}$ corresponding to Lie pseudogroups. In particular, W_n is the np^n-dimensional Jacobson-Witt algebra

$$\mathrm{Der}\{F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)\};$$

for the definition of S_n, H_n, K_n see [7], [8].

Kostrikin and Šafarevič conjectured [7] that every L is of either classical or Cartan type. For rank 1 (where as usual L is said to have rank r if it has a Car-

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tan subalgebra of dimension r) the Kostrikin-Šafarevič conjecture is true by a theorem proved more than 20 years ago by Kaplansky [6], namely, if L has rank 1 then $L \cong sl(2)$ or W_1. An attack on the rank 2 case has seemed particularly important because of the possible implications for general rank both of the result itself (allowing certain modifications involving one root of an independent pair while holding the other fixed, and thus moving from root to root) as well as of whatever methods would be used in its proof. We announce the solution of the rank 2 problem—the result is in accord with the above conjecture.

THEOREM. *Suppose L (finite-dimensional restricted simple over an algebraically closed field of characteristic $p > 7$) has rank 2. Then either L is classical (i.e., of type A_2, C_2 or G_2) or $L \cong W_2$.*

The proof of this result is very long and uses many of the known classification results. We will recall some of these results and then outline the proof of the theorem. Recall that a restricted torus is a restricted subalgebra on which the pth power map is injective, and a toral C.s.a. is a Cartan subalgebra (C.s.a) which is a torus.

Mills and Seligman (see [9]) have characterized the classical L in terms of their Cartan decompositions. A further result (Block [1]) states that (A) if L has a toral C.s.a. such that for each root $\alpha \neq 0$ the root space L_α is one dimensional and $\alpha([L_\alpha, L_{-\alpha}]) \neq 0$, then either L is classical or $L \cong W_1$. Algebras of Cartan type may be characterized in terms of the existence of certain filtrations. Let L_0 be a maximal subalgebra of L, and construct a filtration as follows (see e.g. [10]): Let $L_{-1} \supset L_0$ be an (ad L_0)-submodule of L such that L_{-1}/L_0 is irreducible, $L_i = [L_{i+1}, L_{-1}] + L_{i+1}$ for $i < -1$, and $L_{i+1} = \{x \in L_i \mid [x, L_{-1}] \subset L_i\}$ for $i \geqslant 0$. Then results of Kostrikin and Šafarevič [8], Kac [4], [5] and Wilson [11] state that (B) if L_0/L_1 is a direct sum of classical simple algebras and its center, and if $L_1 \neq 0$ then either L is classical or of Cartan type. In particular, if $L_0/L_1 \cong gl(2)$, $L_1 \neq 0$ and L has rank 2, then either L is classical or $L \cong W_2$.

In practice the application of (B) is difficult since an arbitrary maximal subalgebra L_0 need not yield a filtration for which the hypotheses of (B) hold. The proof of the theorem consists in showing that for a suitable choice of L_0 the hypotheses of (B) are almost always satisfied, and if they are not satisfied for some choice of L_0, then the hypotheses of (A) are satisfied. We now give a sketch of this proof. Let H be a two-dimensional C.s.a. of L. Then H must be a torus, for otherwise H would contain a one-dimensional maximal torus T, contradicting a result of Wilson [12]. Let $L = H + \Sigma_{\gamma \in \Delta} L_\gamma$ be the Cartan decomposition of L with respect to H and let Γ be the subgroup of H^* generated by Δ. Also let $\Gamma_P = \{\gamma \in \Gamma \mid \gamma \neq 0, \gamma([L_{i\gamma}, L_{-i\gamma}]) = 0$ for some i, $1 \leqslant i \leqslant p-1\}$ and write $n(L, H) = |\Gamma_P|/(p-1)$. We say that H *is an optimal* C.s.a. of L if for every two-dimensional toral C.s.a. K of L, $n(L, H) \geqslant n(L, K)$. It can be shown, using results of Winter [13], that if H is optimal then $n(L, H) \geqslant 1$. For $0 \neq \beta \in H^*$ and $0 \neq \gamma \in \Gamma$, define $M_\gamma^\beta = \{x \in L_\gamma \mid \beta([x, L_{-\gamma}]) = 0\}$ and $M^\beta = H + \Sigma_{\gamma \in \Delta} M_\gamma^\beta$. It is easily checked that M^β is a subalgebra. We say that the maximal subalgebra L_0 *is distinguished* if for some optimal H and some $0 \neq \beta \in H^*$ we have $L_0 \supseteq M^\beta$ and either $n(L, H) > 1$ or $n(L, H) = 1$ and $\beta \in \Gamma$, $\beta \notin \Gamma_p$.

We then proceed by induction on dim L. Set $G_0 = L_0/L_1$. Then either the solvable radical solv $G_0 \neq 0$ and so (using Kaplansky's theorem) G_0/solv $G_0 \cong 0$, $sl(2)$ or W_1, or else G_0 is semisimple and hence may be determined using Block's description [2] of semisimple algebras in terms of simple algebras and the induction assumption (as dim $G_0 <$ dim L). A detailed case by case analysis now shows that if L_0 is distinguished then either (B) applies or else $L_1 = 0$. But if $L_1 = 0$ for all choices of a distinguished maximal subalgebra L_0, we can show that (A) must apply, giving the result.

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